On Equisingularity of Families of Maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$

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Abstract. A classical theorem of Briançon, Speder and Teissier states that a family of isolated hypersurface singularities is Whitney equisingular if, and only if, the μ^*-sequence for a hypersurface is constant in the family. This paper shows that the constancy of relative polar multiplicities and the Euler characteristic of the Milnor fibres of certain families of non-isolated singularities is equivalent to the Whitney equisingularity of a family of corank 1 maps from n-space to $n+1$-space. The number of invariants needed is $4n-2$, which greatly improves previous general estimates.

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1. Introduction

Given a family of maps it is useful to have conditions that imply that the family is, in some sense, trivial. Suppose that we have a family of complex hypersurfaces such that each member has an isolated singularity. The family is called *Whitney equisingular* if the singular set of the variety formed by the whole family is a stratum in a Whitney stratification. This implies, for example, that, for each pair of hypersurfaces, there is a homeomorphism between the ambient spaces that takes one hypersurface to the other.

An overall aim of the theory is to find invariants of the elements of the family, the constancy of which implies, or is equivalent to, this Whitney equisingularity. In the isolated hypersurface case the constancy of the μ^*-sequence of Teissier (see for example [13]) is equivalent to the Whitney equisingularity of the family.

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One can consider what happens for maps, rather than varieties, i.e when is a family of complex analytic maps $F : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^p \times \mathbb{C}$ equisingular? There is again a notion of Whitney equisingularity. This time the parameter spaces $\{0\} \times \mathbb{C}$ in $\mathbb{C}^n \times \mathbb{C}$ and $\{0\} \times \mathbb{C}$ in $\mathbb{C}^p \times \mathbb{C}$ have to be strata in a stratification of the map.

In a series of papers Gaffney, (and also in conjunction with others such as Gassler and Massey) has produced some sterling work in answering this question for families where the members have an isolated instability at the origin. In [1] he introduced some new invariants, and proved that the constancy of these invariants within a family is equivalent to Whitney equisingularity.

However, the invariants are very difficult to manipulate, even in low dimensional cases, as the number of invariants is quite large and the method of description varies greatly from one invariant to another. For $\mathbb{C}^2$ to $\mathbb{C}^3$ we need 10 invariants; for $\mathbb{C}^3$ to $\mathbb{C}^4$ we need 20. At present there is no formula in the literature that allows one to calculate the number required.

Nonetheless, because of relationships between them, it is possible to reduce considerably the number of invariants needed in each case. For example, for a family of corank 1 maps from $\mathbb{C}^2$ to $\mathbb{C}^3$ the constancy of only one invariant is required. It should be noted that there was a significant amount of investigation done in [1] to show that this really is the only invariant needed. The $\mathbb{C}^3$ to $\mathbb{C}^4$ case is tackled in [8] where the 20 invariants are reduced to only 8. (The definition of 'reducing' is somewhat vague; one could reduce to one invariant merely by adding together all these upper semi-continuous invariants. The heuristic requirement is that the invariants should be calculable and that they should not be decomposable into other ones.)

The main result (Theorem 3.3) is that we can use relative polar multiplicities and the Euler characteristic of hypersurfaces to produce a Whitney equisingularity result in the case of $p = n+1$, i.e. the image of F is a hypersurface, and where the stable singularities of F have corank 1, (i.e. the differential of F at these points is, at worst, corank 1). So, in particular, the theorem holds when $n \leq 5$.

We reduce the number of invariants to $4n-2$, which is a considerable saving, when n is large (which here means bigger than 3). This saving is achieved, not through using Gaffney's work in [1], but his subsequent work with Gassler, [3], and Massey, [2].

2. Notation and Basic Definitions

In this section we give the definitions related to equisingularity for the sets and the complex analytic maps that concern us, and we reproduce the definitions of two sequences from [3], which in the main theorem will be used to control equisingularity.

Standard definitions from Singularity Theory, such as finite $\mathcal{A}$-determinacy, can be found in [16]. A differentiable map is called *corank 1* if its differential has corank at most 1 at all points.

Often we shall need to move from a germ and choose a representative, or a smaller neighbourhood, etc. Since this is entirely standard and is obvious when it occurs, no explicit mention shall be made of the details as they will be distracting to the exposition.

Definition 2.1. *Let X be complex analytic set and Y a subset of X. We say that X is* Whitney equisingular along Y *if Y is a stratum of some Whitney stratification of X.*

This has been the subject of considerable investigation, see [2] for a survey. Gaffney has studied the notion for the more general case of maps, see [1]. We recall Thom's condition A_f before stating the definition of Whitney equisingularity.

Definition 2.2. *Let $f : X \to Y$ be a complex analytic map. Two strata A and B of a Whitney stratification of X are said to satisfy the* Thom A_f *condition with respect to f at a point $p \in B$ if the differential df has constant rank on A and for any sequence of points $p_i \in A$ such that p_i converges to p and $\ker d_{p_i}(f|_A)$ converges to some T (in the appropriate Grassmannian), then $\ker d_p(f|_B) \subseteq T$. We say f satisfies the* Thom A_f *condition if all pairs of strata satisfy the condition.*

Example 2.3. *Let $f : X \to Y$ be a finite complex analytic map such that X and Y are Whitney stratified so that strata map to strata by local diffeomorphisms. Then, f satisfies the Thom A_f condition as the kernels are all $\{0\}$.*

Now, the main definition is given.

Definition 2.4. *Let $F : (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^p \times \mathbb{C}, 0 \times 0)$ be a family of maps $F(x,t) = (f_t(x), t)$ such that each $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ has an isolated instability at the origin, (i.e. each f_t is finitely $\mathcal{A}$-determined).*

We say that F is Whitney equisingular *if $\mathbb{C}^n \times \mathbb{C}$ and $\mathbb{C}^p \times \mathbb{C}$ can be Whitney stratified so that*

(i). *F satisfies Thom's A_F condition, and*
(ii). *the sets $S = \{0\} \times \mathbb{C} \subseteq \mathbb{C}^n \times \mathbb{C}$, and $T = \{0\} \times \mathbb{C} \subseteq \mathbb{C}^p \times \mathbb{C}$ are strata. (That is the 'parameter axes' are strata.)*

Remark 2.5. *This means, by the Thom-Mather Second Isotopy Lemma, that the members of the family are topologically equivalent.*

Let $f : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ be a complex analytic function, and denote the Jacobian ideal by $J(f)$:

$$J(f) = \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_N} \right)$$

for coordinates $z_1, \ldots, z_N$ in $\mathbb{C}^N$.

Definition 2.6. *The* blowup of $\mathbb{C}^N$ along the Jacobian ideal, *denoted $Bl_{J(f)}\mathbb{C}^N$, is the closure in $\mathbb{C}^N \times \mathbb{P}^{N-1}$ of the graph of the map*

$$\mathbb{C}^N \backslash V(J(f)) \to \mathbb{P}^{N-1}, \qquad x \mapsto \left(\frac{\partial f}{\partial z_1}(x) : \cdots : \frac{\partial f}{\partial z_N}(x) \right),$$

where $V(J(f))$ is zero-set of $J(f)$.

A hyperplane h in $\mathbb{P}^{N-1}$ can be pulled back by the natural projection $p : \mathbb{C}^N \times \mathbb{P}^{N-1} \to \mathbb{P}^{N-1}$ to a Cartier divisor, H, on $Bl_{J(f)}\mathbb{C}^N$, (provided $Bl_{J(f)}\mathbb{C}^N$ is not contained in the product of $\mathbb{C}^N$ and h). We call this a *hyperplane on* $Bl_{J(f)}\mathbb{C}^N$.

Let $b : \mathbb{C}^N \times \mathbb{P}^{N-1} \to \mathbb{C}^N$ be the other natural projection. For suitably generic hyperplanes $h_1, \ldots, h_k$ in $\mathbb{P}^{N-1}$, the multiplicity at the origin of $b(H_1 \cap \cdots \cap H_k \cap Bl_{J(f)}\mathbb{C}^N)$ is an analytic invariant of $V(J(f))$, see [3].

Definition 2.7. *The* kth relative polar multiplicity of f *is the multiplicity of the variety* $b(H_1 \cap \cdots \cap H_k \cap Bl_{J(f)}\mathbb{C}^N)$ *at the origin. It is denoted by* $m_k(f)$.

Remark 2.8. *Full details of this construction and proofs of the various assertions can be found in* [3] *where the authors also show that the situation can be generalised to ideals other than the Jacobian.*

We can now define another sequence of invariants; again these have a topological nature.

Definition 2.9 ([2] **p238).** *Let* $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ *be a complex analytic function and* $L_i \subseteq \mathbb{C}^{n+1}$ *be a generic* i*-dimensional linear subspace. Denote the Euler characteristic of the Milnor Fibre of* $f|L_i$ *by* $\chi^i(f)$.

From this we can define a sequence

$$\chi^*(f) := \left(\chi^{n+1}(f), \ldots, \chi^2(f)\right).$$

In the case of an isolated singularity, this (effectively) reduces to the standard μ^*-sequence in Equisingularity Theory.

Remark 2.10. *It transpires that the number* $\chi^1(f)$ *is not needed in the theory in* [3] *and so is omitted.*

Example 2.11. *If* f *defines the Swallowtail singularity, (i.e. the image of the stable map* $(x, y, z) \mapsto (x, y, z^4 + xz^2 + yz)$*), then* $\chi^3(f) = 1$*, (see, for example,* [11] *page 54), and* $\chi^2(f) = 6$*. The latter can be calculated using a program such as* `Singular`.

In general, it is not known how to calculate the homology of the Milnor Fibre of a non-isolated singularity. In some cases it is possible to calculate the Euler characteristic in practice, for example, using Massey's theorem that it is equal to the alternating sum of the Lê numbers, see [11].

3. Main Theorems

Let $X \subseteq \mathbb{C}^N \times \mathbb{C}$ be a family of hypersurface germs defined by $H : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, and where $H(x, t) = (h_t(x), t)$ and Y is the parameter stratum $\{0\} \times \mathbb{C} \subseteq \mathbb{C}^N \times \mathbb{C}$.

We can now tease out the important elements of the proof of Theorem 6.6 of [2] to prove the following.

Lemma 3.1. *Suppose that $X \backslash Y$ is Whitney stratified such that, at each point of a stratum in $X \backslash Y$, X is locally analytically a product of a normal slice and the stratum. Suppose further that the complex link of every stratum in $X \backslash Y$, and not in the non-singular part of X, is not contractible.*

Then, the family is Whitney equisingular along Y if, and only if, the sequence

$$\left(m_{N-1}(h_t), \ldots, m_1(h_t), \chi^N(h_t), \ldots, \chi^2(h_t)\right)$$

is independent of $t \in Y$.

Proof. Suppose that the sequence is constant. Theorem 6.5 of [2] states that the sequence being constant in the family implies that the non-singular part of X is Whitney regular along Y.

We can now deal with the strata in the singular part of X. Suppose that R is a stratum of X of dimension r. Take the normal slice to R at the point p, i.e., the set $M \cap X$ where M is a manifold transverse to R with $M \cap R = \{p\}$. Since X has a product structure we can assume that (M, p) is $(\mathbb{C}^{N-r}, p)$. Then, $M \cap X$ will be a hypersurface defined locally at p by the germ $g : (\mathbb{C}^{N-r}, p) \to (\mathbb{C}, 0)$ say.

By definition, the complex link of the stratum R is the complex link of $M \cap X$ at the point p. This complex link is homotopically equivalent to a wedge of spheres (since the space is a hypersurface, see [4] p187), the number of which is the multiplicity of the relative polar curve of g, see Massey [10] page 365. Since, by assumption, this number is positive, the polar curve is non-empty. The example on page 235 of [2] shows that this implies that the origin of $\mathbb{C}^{N-r}$ is the image of a component of the exceptional divisor of $Bl_{J(g)}\mathbb{C}^{N-r}$. Since X has an analytic product structure along R this means that the closure of R is the image of a component of the exceptional divisor of $Bl_{J(H)}(\mathbb{C}^N)$. Thus, by the assumptions of the statement of the lemma and by using Theorem 6.5 of [2], we conclude that R is Whitney regular along Y.

The converse is just Theorem 6.3 of [3]. $\square$

It seems likely that requiring that the complex links are non-contractible is necessary. This is because the topology of functions is intimately connected with complex links. (In [14], Tibăr shows that the Milnor fibre of a function with an isolated singularity on a complex analytic set is homotopically equivalent to a bouquet of suspensions of the complex links of the strata of the set.)

There are not many general results on the non-contractibility of complex links, see Section 4 of [15] for examples of hypersurfaces with a stratum that has a contractible complex link and for a theorem that states that complete intersections with a singular locus of dimension less than 2 have non-contractible complex links.

We can use the above lemma to prove our main theorem. First we need a definition:

Definition 3.2. *Let $p : A \to B$ be a continuous map. Then the* double point space of p in the source *is the set*

$$\text{closure}\,\{a \in A \,|\, \textit{there exists } a' \in A \textit{ such that } p(a) = p(a'),\ a \neq a'\}\,.$$

For a finitely $\mathcal{A}$-determined map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ the double point space is a hypersurface in $\mathbb{C}^n$.

Theorem 3.3. *Let $F : (\mathbb{C}^n \times \mathbb{C}, 0 \times 0) \to (\mathbb{C}^{n+1} \times \mathbb{C}, 0 \times 0)$ be a 1-parameter family of finitely $\mathcal{A}$-determined map-germs whose locus of instability is $\{0\} \times \mathbb{C} \subseteq \mathbb{C}^{n+1} \times \mathbb{C}$.*

Suppose that the stable singularity types appearing in F are corank 1 (e.g., $n \leq 5$ or each f_t is corank 1).

Then, F is Whitney equisingular if, and only if,

$$\big(m_n(h_t), \dots, m_1(h_t), \chi^{n+1}(h_t), \dots, \chi^2(h_t)\big) \text{ and}$$
$$\big(m_{n-1}(g_t), \dots, m_1(g_t), \chi^{n}(g_t), \dots, \chi^2(g_t)\big)$$

are independent of $t \in Y$, where h_t is the function defining the image of f_t, and g_t is the function defining the double point set in the source of f_t.

Proof. Since the stable types are corank 1 and the map F is stable outside the instability locus of the members, we can stratify the source and target by stable type to get a Whitney stratification outside S and T (the parameter axes), such that the double point set in the source is a union of strata. (The stratification is finite as there are only finitely many right-left equivalence classes of stable germs of corank 1.) Furthermore, this stratification by stable types means that locally the spaces have a product structure – the one arising from unfolding minimal stable maps. Also, since F is finite, the strata map to strata by a local diffeomorphism. Thus, outside S and T the map is Thom A_F, see Example 2.3.

Theorem 7.3 of [6] states that the complex link of a stratum of the image of a corank 1 stable map is homotopically equivalent to a single sphere (except for the 'top' stratum which is the non-singular part of the image and hence has an empty complex link). Thus, we can apply Lemma 3.1 to the family of hypersurfaces giving the image of F to show that the image of F is Whitney equisingular along T.

Now, the double point set of F is also a family of hypersurfaces. Furthermore, outside S, it is also the image of a stable corank 1 map, see Proposition 3.5.1 of [5]. Thus, again applying Lemma 3.1, the double point set is Whitney equisingular along S.

Since S maps to T and this map is a local diffeomorphism we see that F is Thom A_F. Since source and target are Whitney stratified we conclude that F is Whitney equisingular. □

Thus we can reduce the number of invariants required to $4n - 2$ invariants. This is a considerable saving. For example, for $n = 2$ we get 6 invariants compared to Gaffney's original 10, and for $n = 3$ we get 10 rather than the orginal 20. Note however, that in the former case Gaffney reduced to 1 invariant and in the latter Pérez reduced the number of invariants to 8. It is in the cases where $n > 3$ that the theorem comes into its own. So far no-one has attempted to tackle the large task of enumerating precisely Gaffney's invariants for $n = 4$ or the even greater task of reducing through utilising relationships between them.

Whilst at the meeting in Luminy Marcelo Saia informed me that, in [9], Jorge Pérez and Saia show how the number of Gaffney's original invariants can be cut, more or less, in half for corank 1 maps. This is rather suggestive as, in a similar vein, Corollary 8.8 of [1] states that, for maps in the theorem with $n = 2$, the map F is Whitney equisingular if its *image* is Whitney equisingular along the parameter axis. Combining this observation with the result in [9] we can conjecture that the same is true for more general n. If this were the case, then it would imply that the theorem above could be improved further as we could drop the assumption concerning the sequences associated with the double point set, i.e. we would require only the $2n$ invariants controlling the Whitney equisingularity of the image and could discard those in the source.

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