Black and White Local Invariants of Mappings from Surfaces into Three-Space

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Abstract

Goryunov proved that the space of local invariants of Vassiliev type for generic maps from surfaces to three-space is three-dimensional. The basic invariants were the number of pinch points, the number of triple points and one linked to a Rokhlin type invariant. In this paper we show that, by colouring the complement of the image of the map with a chess board pattern, we can produce a six-dimensional space of local invariants. These are essentially black and white versions of the above. Simple examples show how these are more effective.

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1 Introduction

In [1] Goryunov showed the existence of a set of local invariants for generic maps $f : N \to \mathbb{R}^3$ for a fixed closed and orientable surface N. See also the announcement made in [2]. The invariants found were generated by the number of pairs of pinch points, the number of triple points and an invariant connected to a Rokhlin type invariant. To prove that the space of local invariants (under the assumptions made) was three-dimensional Goryunov used a Vassiliev type method to study the discriminant (the set of non-generic maps) in the space of all smooth maps from N to $\mathbb{R}^3$.

The purpose of this paper is to show how to produce finer invariants by using the existence of a chess board colouring of the complement of the image of f in $\mathbb{R}^3$. This involves defining each region in the complement of the image to be either black or white such that neighbouring regions, i.e. those sharing a face, are different colours. This allows us to produce pairs of invariants, a black and a white one in each pair, such that the sum of the black and white pair, or in one case the average, is equal to one of Goryunov's invariants. The precise list of invariants is given in Theorem 4.3 of Section 4.6. The proof of their existence is longer than that of Goryunov's since ignoring the black and white colouring allows one to make some significant simplifications. Another difference is that there are more terms used in calculating the black and white invariants. For example, one has to consider quadruple points, which one does not have to do in Goryunov's situation.

We also study black and white invariants for immersions (Theorem 4.4) and invariants mod 2 (Theorem 4.5). The situation for the latter is not so simple as some of the invariants in Goryunov's lists do not come in straightforward black and white versions. Mod 2 invariants for immersions of spheres are given in Theorem 4.6.

In Section 5 the effectiveness of the black and white invariants is demonstrated in a simple example of two generic maps which Goryunov's invariants are unable to distinguish.

Goryunov's invariants are examples of Vassiliev type invariants of order 1. The order 1 results presented here are necessary for a full understanding of the order 2 case which the author hopes to deal with in a later paper. It seems possible that the results can also be extended to the situation covered in the recent preprints [6] and [7].

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2 Order One Vassiliev Invariants

Let Ω denote the space of all smooth maps, that is C^∞, from a closed 2-manifold N to a 3-manifold P. The set of all unstable maps, in the sense of right-left equivalence, Δ, is of codimension 1 in Ω and so Δ separates Ω into disjoint components, the elements of which are called *generic maps*. The elements of a connected component are all right-left equivalent. An *invariant* of generic maps is a real function on the set of components of $\Omega \setminus \Delta$. If we take a generic path in Ω then we can analyse the invariants by seeing how they change as the path crosses Δ.

The set Δ has a subset of codimension 1 (codimension 2 in Ω), denoted by D, such that $\Delta \setminus D$ is the set of generic degenerate maps. The set $\Delta \setminus D$ contains the elements of Ω with exactly one of the following set of singularities, which are pictured in Figures 2 and 3 of [1] and reproduced here in the middle diagrams in Figure 1:

$$E, H, T, C, B, K, Q.$$

The letter E stands for elliptic, H for hyperbolic, T for triple, C for cross cap, B for bubble, K for cone (in Russian) and Q for quadruple.

The sets in $\Delta \setminus D$ corresponding to these are called the *top strata* of Δ, as in effect they are the beginning of a stratification of Δ. A generic path in Ω will cross Δ only in these strata. If we take a subdivision of the strata we will also call the elements of the subdivision strata. If we can define an unambiguous positive and negative side to the stratum, then we say that the stratum is *coorientable*. In [1] a side of a stratum is *positive* if there is a feature not present on the other side. E.g. the appearance of a local 2-cycle (also known as a bubble) or a cross cap. This works for the E, T, C, B and K strata. Positive crossings of these can be seen in Figure 1. We can define positive crossings for H and Q if N is orientable, and will do so in the next section.

Definition 2.1 *An invariant is an* order one Vassiliev invariant *if the change (also called the* jump*) in the value of the invariant when crossing from one side of a stratum to another is completely determined by the stratum and the direction of crossing with respect to the coorientation of the stratum.*

This type of invariant is named after Vassiliev since he was the first to use this type of method for knot invariants.

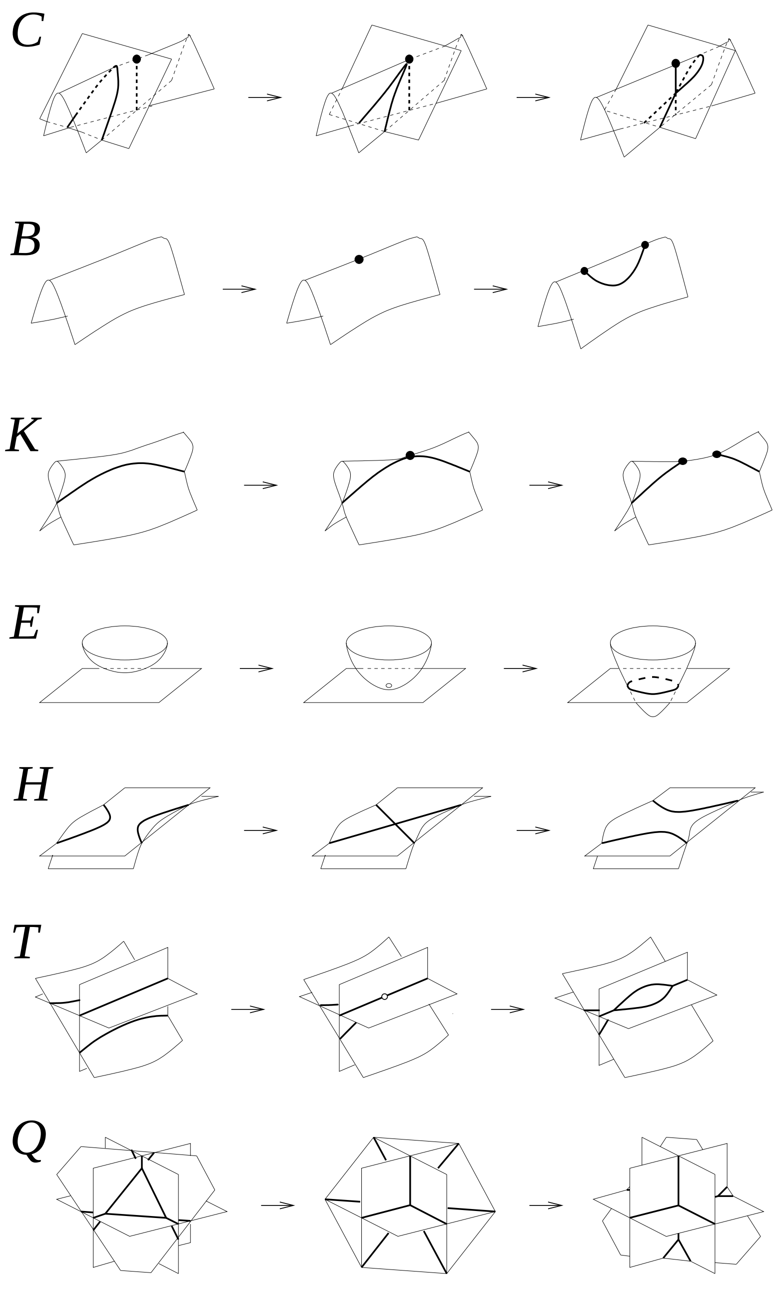

Figure 1: Local bifurcations of codimension 1 singularities. Taken from pictures originally drawn by V.V Goryunov.

If no coorientation exists for a stratum or one is not chosen, then the invariant should not change at that stratum, unless one is dealing with invariants over $\mathbb{Z}_2$ as then coorientation does not matter. This is because if we take a path which crosses a stratum and immediately returns, then the total jump in the invariant should obviously be zero. So for non-cooriented strata a non-zero jump would lead to a jump of two times a constant.

If we take loops in the space $\Omega \setminus D$ then we in effect place conditions on the jumps of our invariants: the total change in the value of an invariant must be zero on any loop in $\Omega \setminus D$. Thus by taking all possible loops up to homotopy in $\Omega \setminus D$ we define a system of linear equations which jumps of an invariant must satisfy. Solving the system gives us invariants up to an additive constant.

To calculate an invariant we define the value on some standard embedding of the surface and then we take a path in $\Omega \setminus D$ from the standard embedding of N to f. The invariant changes only as we cross the top strata.

3 Goryunov's Local Invariants

Suppose now that $P = \mathbb{R}^3$ (so $\Omega = \{f | f : N \to \mathbb{R}^3\}$) and that N is oriented. The orientability of N allows us to choose an unambiguous normal at every immersive point of the image of f such that the choice is compatible over the whole this space. This orientation of the image meant that Goryunov was able to further subdivide the seven top strata of Δ in Ω, so that there are 20 strata:

$$E^0, E^1, E^2, T^0, T^1, T^2, T^3, C^{++}, C^{+-}, C^{-+}, C^{--}, B^+, B^-, K^+, K^-, Q^4, Q^3, H^-$$

and

$$Q^2, H^+.$$

The last two we highlight as they are not coorientable strata. The coorientable strata are pictured in Figure 2.

The superscripts refer to the coorientations of a feature appearing in the local bifurcation. Thus for E^j, T^j, and Q^j, the j counts the number of pieces of the appearing sphere with outward pointing normal.

The signs $+$ and $-$ denote whether a normal is pointing in or out of a feature. For $C^{a,b}$ the sign a denotes the direction of the normal for the pinch point part of the sphere, positive for outwards, and b denotes the same for the plane. Similiarly for B^+ and B^-. For K^+ the normal on the tubular part before the bifurcation points outwards and for K^- it points inwards. The H^+ strata has the normals pointing in coincident directions at the tangency, H^- has opposite normals.

Through the coorientation of the image we can coorient the strata H^-, Q^3, and Q^4 as in Figure 2. However, using this method does not give us a way to define a coorientation for H^+ and Q^2. Subsequently, we shall assume these to have a jump of zero for any invariant.

The main result of [1] is that a classification of codimension 2 singularities leads to 17 independent linear equations. Hence the space of order one Vassiliev invariants has dimension 3. Up to an additive constant these invariants are generated by

$$\begin{aligned} I_p &= B + K, \\ I_t &= 2T + C, \\ I_3 &= E^2 - E^0 + H^- + T + C^{++} + C^{+-} + B^+ + K^+. \end{aligned}$$

Here, B denotes the total number of B^+ and B^- type strata crossed, etc. Goryunov called these local invariants, since they only altered under changes of the topological

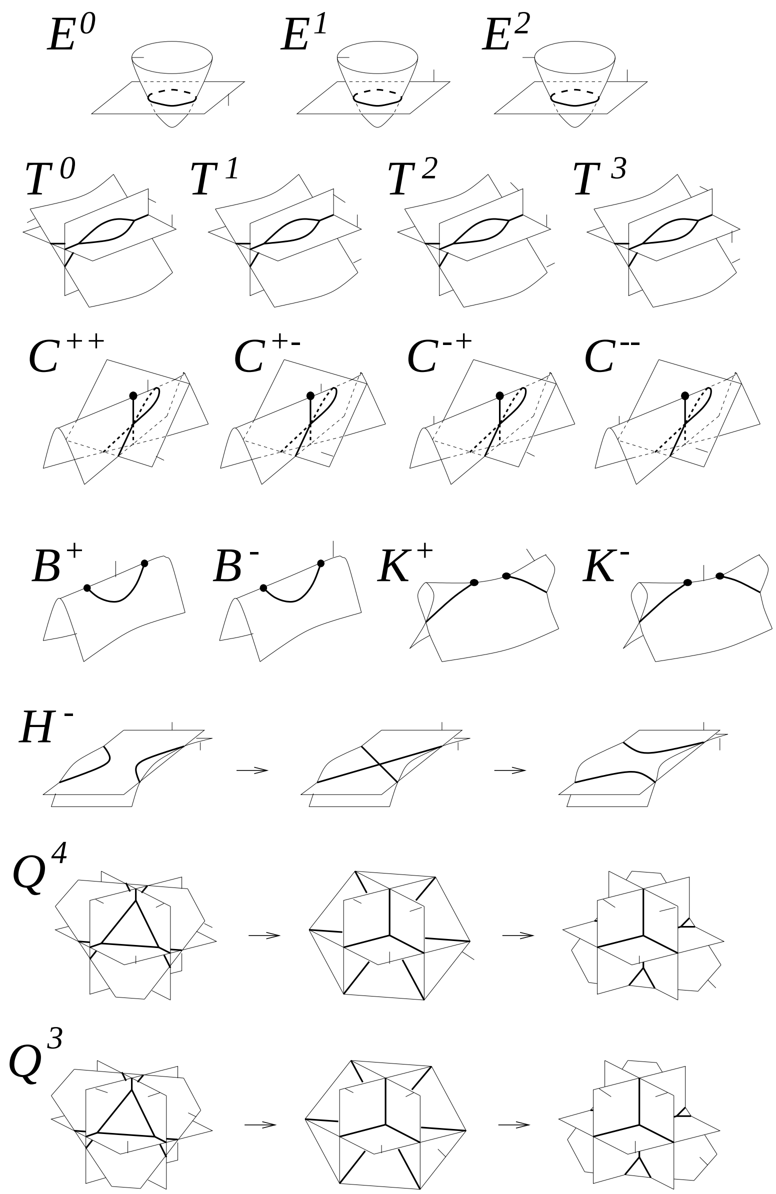

Figure 2: Further subdivision of strata (following [1])

type of the local bifurcation in the image. Note however, that they were defined using the orientability of N, which is a global property. We shall refer to them as *Goryunov invariants.* The intention of this paper is to find black and white versions of these invariants, see Section 4.

The third invariant I_3 is the essentially new one and it is linked to a Rokhlin type invariant. See [1] for full details. If we drop the assumption that N is orientable then the 7 generic pieces in the stratification of Δ can be used to get local invariants. However these are just I_p and I_t; the third invariant depends on orienting N.

The invariants are normalised using the standard embedding of N into $\mathbb{R}^3$ and by choosing an orientation: as N is compact it is homeomorphic to a sphere or an n-holed torus and each of these has a standard embedding into $\mathbb{R}^3$.

From this we can normalise I_p and I_t to be zero on the standard embedding of N. Thus, when we take a generic map f, $I_p(f)$ is the number of pairs of pinch points and $I_t(f)$ is the number of triples. Describing the normalisation of I_3 is more difficult, but it can be normalised to behave well under connected summation of surfaces, see [1] for details.

Example 3.1 *The Euler Characteristic of the image of f is a Goryunov invariant: For a generic map $f : N^2 \to P^3$, with N^2 a closed surface and P^3 a three dimensional manifold, the Euler characteristic of the image is given by*

$$\chi(f(N)) = \chi(N) + \frac{C(f)}{2} + T(f)$$

where now $C(f)$ is the number of cross caps and $T(f)$ is the number of triple points, see [4]. So we see that

$$\chi(f(N)) = \chi(N) + I_p + I_t.$$

Since $\chi(N)$ is in effect an additive constant and I_p and I_t are order one Vassiliev invariants we see that $\chi(f(N))$ is an order one Vassiliev invariant too.

4 Black and White Goryunov Invariants

We now come to the main part of this paper. We assume we are dealing with generic maps $f : N \to \mathbb{R}^3$ with N oriented as before. Using another global property of the map it is possible to further refine the stratification of $\Delta \setminus D$ and obtain a set of 6 order one invariants. This is done using a chess board colouring of the complement of the image in the ambient space.

The complement of the image of f in $\mathbb{R}^3$ is the union of a number of disjoint subsets of $\mathbb{R}^3$. Each of these pieces can be assigned a unique colour. It is well known that using two colours, black and white, we can colour the complement of the image so that regions that are adjacent (separated by a piece of plane) are different colours. This is a three-dimensional version of a chess board pattern. Since N is compact we can choose a colour for the region 'at infinity'.

The colouring of the complement allows us to assign colours to cross caps and triple points. So we expect to find local invariants that count the number of black cross caps, etc., and this indeed turns out to be the case.

4.1 Black and White Cross Caps

A cross cap's colour is defined to be the colour in the region that is 'pinched' by the line of self intersection. An example is shown in Figure 3.

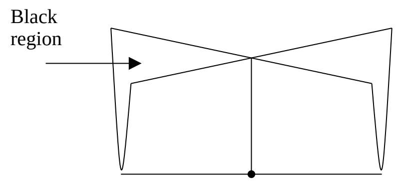

Figure 3: A black cross cap

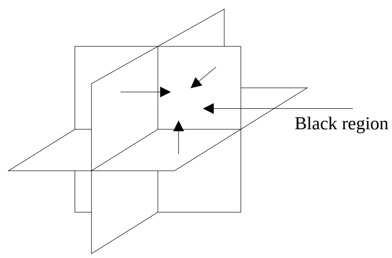

Figure 4: A black triple point

4.2 Black and White Triple Points

For each triple point there exist eight nearby regions in the complement of the image. The colour of the triple point is defined as the colour of the region for which all the normals to the image arising from the orientation of N point inwards. An example is shown in Figure 4.

4.3 Refining the Stratification of $\Delta \setminus D$

The stratification of Δ is further refined by defining a stratum to be black or white. Denote the colour of the stratum by a subscript B or W, for Black or White. For E, T, C, B and Q we use the fact that each positive side contains a 2-cycle, called a bubble, and the stratum is coloured according to the colour of the region enclosed in the bubble. For example C_B^{++} denotes the substratum of C^{++} where there is a black bubble on the positive side. (In the case of the Q stratum the bubble is the same colour on both sides of the stratum.)

The stratum H also has an enclosed region, (though in this case it is not a bubble), and is refined by the colour in the region.

The stratum K is refined using black and white depending on the colour of the cross caps created. For example, K_B^+ denotes the stratum where two black cross caps are created on the positive side.

Definition 4.1 *A* black and white local invariant *is an order one Vassiliev invariant with respect to the above black and white stratification of the discriminant.*

4.4 Tables

When crossing strata it is helpful to know the number of black/white cross caps and triple points that appear. For instance, crossing the strata $K_B^\pm$ or $B_B^\pm$ in the positive direction produces 2 black cross caps. Similarly for the white versions. The following tables give the information for positive crossing of C, T and Q strata.

Colour of the pair of triples appearing when crossing T strata:

	T^0	T^1	T^2	T^3
Black Bubble	B	W	B	W
White Bubble	W	B	W	B

For example, passing through T^1_W produces 2 black triples.

Colour of triple after crossing $C^{**}_{B/W}$:

	C^{++}	C^{+-}	C^{-+}	C^{--}
Black Bubble	W	B	W	B
White Bubble	B	W	B	W

For example, passing through C^{-+}_B produces a white triple point. Obviously for $C^{**}_{B/W}$ the pinch point is B/W.

Crossing Q strata:

	B on negative side	W on neg.	B on positive	W on pos.
Q^4_B	4	0	0	4
Q^3_B	1	3	3	1
Q^4_W	0	4	4	0
Q^3_W	3	1	1	3

For example, if we pass through Q^3_W with 1 black and 3 white triples on the positive side, then on the negative side we have 3 black and 1 white triple points.

If we pass through any Q^2 stratum, then the number of black and white triples does not change. (Hence we cannot use black/white colouring to coorient the Q^2 strata).

4.5 System of Equations

Goryunov's paper relied on a classification of $\mathcal{A}_e$-codimension 2 multi-germs. Since the appearance of his paper a good reference for this list has been made available, see Hobbs and Kirk, [3]. From this list we take our notation for multi-germ singularities and use the standard Mond notation for mono-germs, see [5]. The precise details of the Hobbs and Kirk notation is given on pages 66-67 of [3] and is not especially relevant to us except for naming purposes. Briefly, a plane is written as A_0 and the cross cap is denoted S_0. Two singularities Q and P with transverse intersection is written QP; non-transverse intersection with $A^{\pm}_k$-contact is written as $QP|A^{\pm}_k$.

Using the classification of codimension 2 singularities of [3] we can prove that there are 34 independent equations, which come in black and white pairs. Since each top stratum is divided into black and white we double the number of top strata to 40. If we ignore the colours, then we just get Goryunov's result.

In the following, the precise verbal descriptions will be clarified by subsequent diagrams. The codimension 2 singularities are:

Mono-germs

- A_2, is given by $(x, y) \mapsto (x, y^2, y(y^2 + x^3))$,
- $B^{\pm}_2$, is given by $(x, y) \mapsto (x, y^2, y(x^2 \pm y^4))$,
- H_2, is given by $(x, y) \mapsto (x, y^3, xy + y^5)$.

Bi-germs

- $A_0^2|A_2$, degenerate tangency of two smooth sheets,
- $(A_0S_0)_2$, interaction of curved plane and double tangency of double line in cross cap,
- $A_0S_1^{\pm}$, interaction of plane and B or K singularities,
- $A_0S_0|A_1^{\pm}$, curved plane and cross cap.

Tri-germs

- $A_0^3|A_2$, two tranverse planes and degenerate tangency of third sheet,
- $(A_0^2|A_1^{\pm})(A_0)|A_1$, two transverse planes and curved third sheet, (elliptic or hyperbolic),
- $A_0^2S_0|A_1$, two transverse sheets and a cross cap.

Quadri-germ

- $(A_0^3|A_1)(A_0)$, triple point and curved sheet.

Quinti-germ

- A_0^5, five planes interacting.

All but $A_0S_1^{\pm}$ are accounted for in [1] but the 8 equations arising from these singularities are trivial in the system there.

In the following list of equations the jump on the stratum S is denoted by s.

Lemma 4.2 *Loops in the space $\Omega \setminus D$, i.e. around codimension 2 singularities, give the following 34 independent equations.*

$$b^+_{B/W} - k^+_{B/W} = 0 \tag{1}$$

$$b^-_{B/W} - k^-_{B/W} = 0 \tag{2}$$

$$k^+_{B/W} - h^-_{B/W} - b^-_{B/W} = 0 \tag{3}$$

$$b^+_{B/W} + c^{--}_{W/B} - c^{++}_{B/W} - b^-_{W/B} = 0 \tag{4}$$

$$e^2_{B/W} - h^-_{B/W} = 0 \tag{5}$$

$$e^0_{B/W} + h^-_{B/W} = 0 \tag{6}$$

$$e^1_{B/W} - h^+_{B/W} = 0 \tag{7}$$

$$c^{--}_{W/B} - t^3_{B/W} + c^{++}_{B/W} = 0 \tag{8}$$

$$c^{-+}_{W/B} - t^2_{B/W} + c^{+-}_{B/W} = 0 \tag{9}$$

$$c^{+-}_{W/B} - t^1_{B/W} + c^{-+}_{B/W} = 0 \tag{10}$$

$$c^{++}_{W/B} - t^0_{B/W} + c^{--}_{B/W} = 0 \tag{11}$$

$$-c^{-+}_{B/W} - e^2_{B/W} + c^{++}_{B/W} = 0 \tag{12}$$

$$-c^{--}_{B/W} + e^0_{B/W} + c^{+-}_{B/W} = 0 \tag{13}$$

$$c^{-+}_{W/B} + t^2_{B/W} + c^{++}_{B/W} + q^3_{W/B} - c^{-+}_{B/W} - t^3_{B/W} - c^{++}_{W/B} = 0 \tag{14}$$

$$c^{--}_{W/B} + t^1_{B/W} + c^{++}_{B/W} + q^4_{W/B} - c^{--}_{B/W} - t^2_{B/W} - c^{++}_{W/B} = 0 \tag{15}$$

$$h^+_{B/W} = 0 \tag{16}$$

$$q^2_{B/W} = 0 \tag{17}$$

The equations come in black and white pairs, so for instance, (8) represents the two equations

$$c_W^{--} - t_B^3 + c_B^{++} = 0$$

and

$$c_B^{--} - t_W^3 + c_W^{++} = 0.$$

One of the main differences between the equations here and in [1] is that the equations (14) and (15) arising from the third tri-germ (this involves the Q^3 and Q^4 strata) do not give $q^3_{B/W} = q^4_{B/W} = 0$. Note also that the simplifications in Goryunov's paper, such as $t^0 = t^1 = t^2 = t^3$, do not hold for our set of equations, and thus we cannot just copy Goryunov's proof making only minor modifications.

Proof. The bifurcation diagrams that lead to the equations can be found in [1] and [3]. The last two pairs of equations in the list follow from the non-coorientability of the strata H^+ and Q^2. The germs that we use in the above equations are A_2, B_2^-, H_2, $A_0^2|A_2$, $(A_0S_0)_2$, $A_0S_0|A_1^+$, and $A_0^2S_0|A_1$.

Mono-germs. The singularity A_2 gives rise to equations (1) and (2). The third is given by B_2^- which is then combined with (5) to prove the triviality of the equation arising from B_2^+. The last mono-germ H_2 gives (4). See Figure 5.

Bi-germs. For the bifurcation diagrams for bi-germs see Figure 6. The next three pairs of equations are deduced from $A_0^2|A_2$, the degenerate tangency of two smooth sheets. The singularity $(A_0S_0)_2$, the interaction of a curved plane and tangency of the double point line in a cross cap, gives equations (8) to (11).

The equations arising from $A_0S_1^\pm$, the interaction of a plane and the B or K codimension 1 singularities, are trivial in the system. The equations for $A_0S_1^+$ are

$$b^*_{B/W} + c^{*\pm}_{W/B} - c^{*\mp}_{B/W} - b^*_{W/B} = 0.$$

These can be obtained from (1) to (6), (12) and (13). The equations for $A_0S_1^-$ are just those for $A_0S_1^+$ with b's replaced by k's. They can be obtained using (1) and (2).

The singularity $A_0S_0|A_1^+$, the interaction of a curved plane and a cross cap, gives equations (12) and (13), (this also uses (7) and (16)). The equations for the dual singularity, $A_0S_0|A_1^-$, can be deduced from these in conjunction with (5), (6) and (7).

Tri-germs. The first two tri-germs, $A_0^3|A_2$ and $(A_0^2|A_1^\pm)(A_0)|A_1$ give rise to trivial equations:

The former is the interaction of the transverse crossing of 2 planes and the degenerate tangency of a third sheet to the line of self intersection, see Figure 7. The resulting equations involve only the T strata:

$$t^0_{W/B} - t^3_{B/W} = 0 \quad (18)$$

$$t^2_{W/B} - t^1_{B/W} = 0 \quad (19)$$

and these are trivial using (8) to (11).

The latter singularity comes in a positive and a negative form. One is the interaction of an elliptic sheet with two planes crossing transversely. The negative one is with a hyperbolic sheet. Of the eight possible black/white pairs of equations possible from the orientations of three sheets only three of the positive singularity are of consequence:

$$e^2_{B/W} + t^2_{W/B} - t^3_{B/W} - e^2_{W/B} = 0 \quad (20)$$

$$e^1_{B/W} + t^1_{W/B} - t^2_{B/W} - e^1_{W/B} = 0 \quad (21)$$

$$e^0_{B/W} + t^0_{W/B} - t^1_{B/W} - e^0_{W/B} = 0. \quad (22)$$

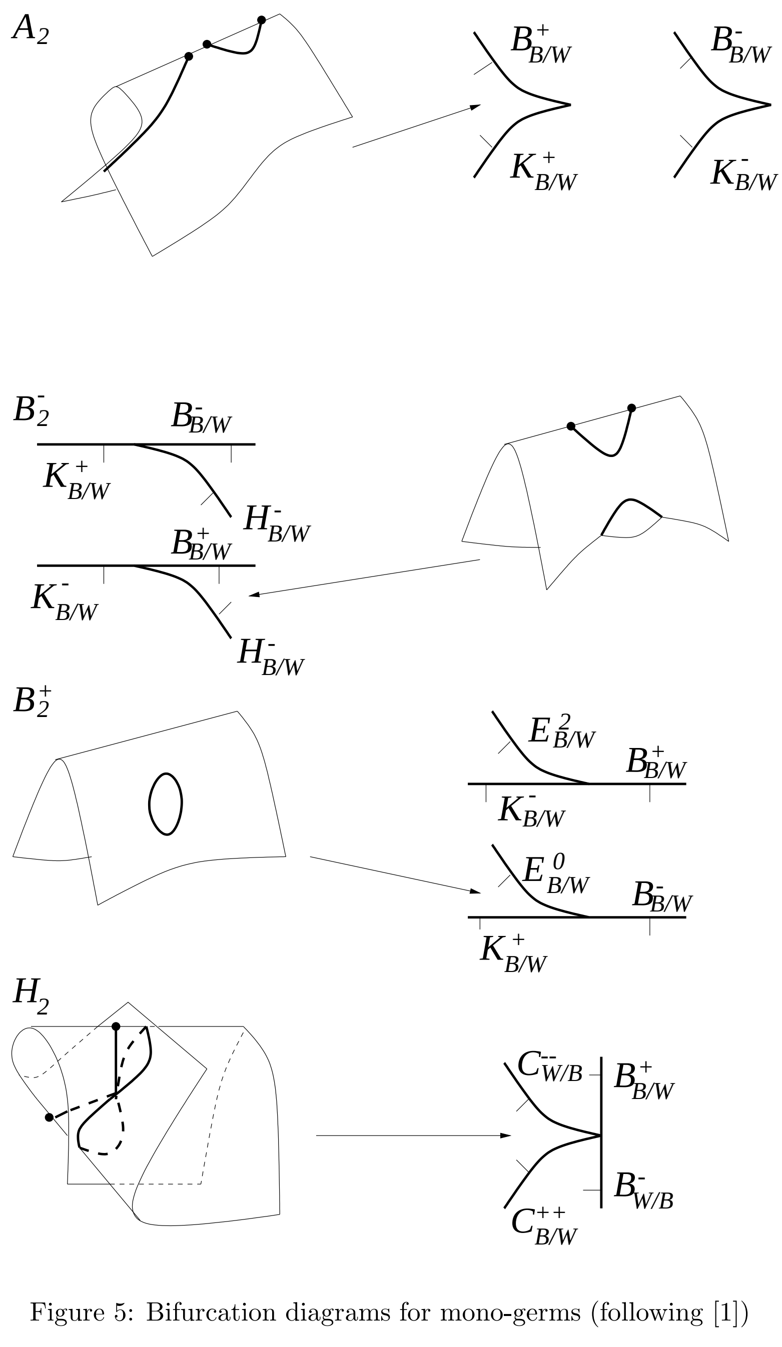

Figure 5: Bifurcation diagrams for mono-germs (following [1])

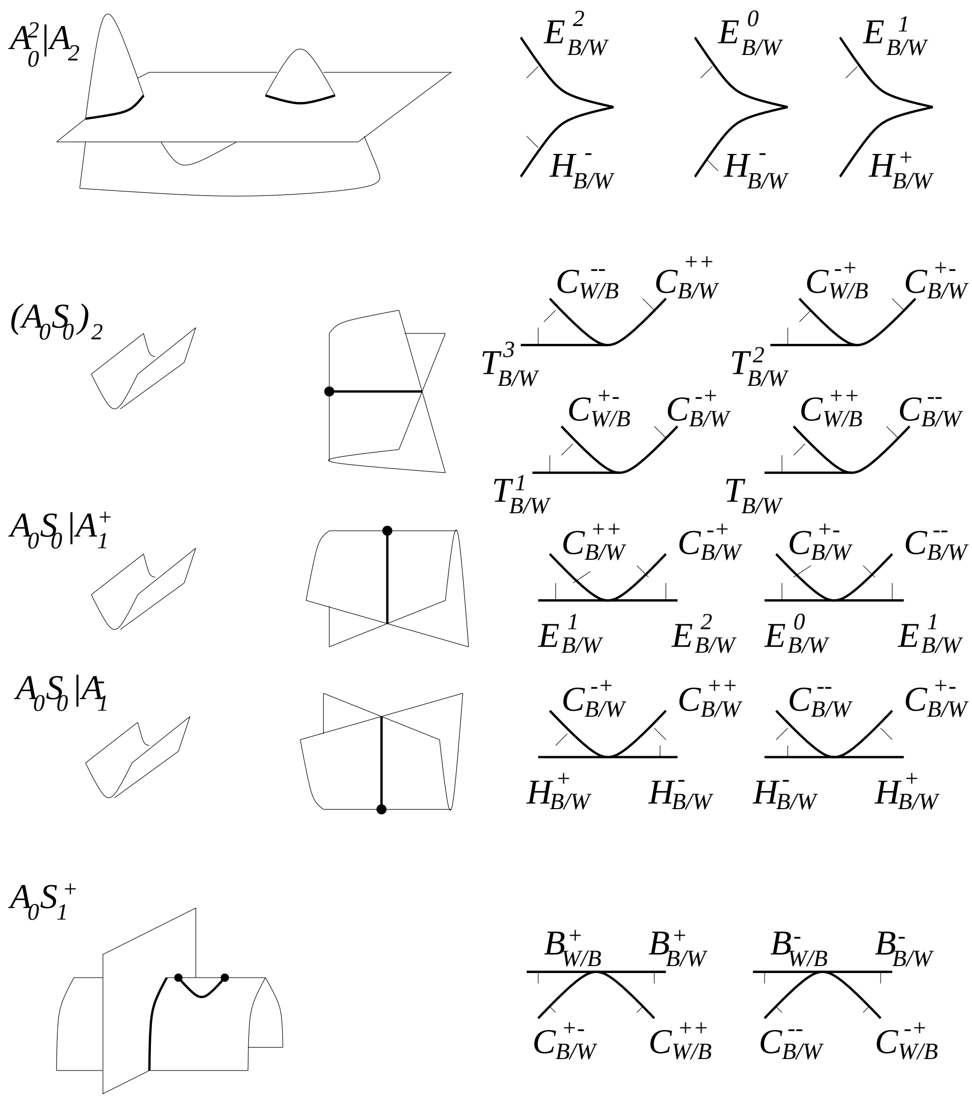

Figure 6: Bifurcation diagrams for bi-germs (following [1])

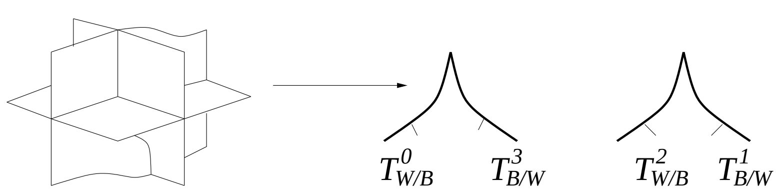

Figure 7: Bifurcation diagrams for $A_0^3|A_2$

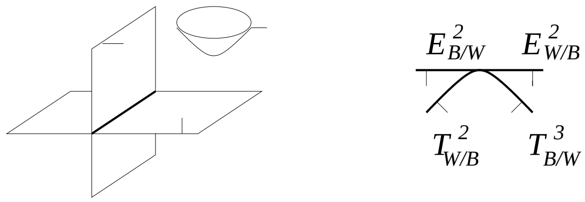

Figure 8: Bifurcation diagram for $(A_0^2|A_1^+)(A_0)|A_1$ (following [1])

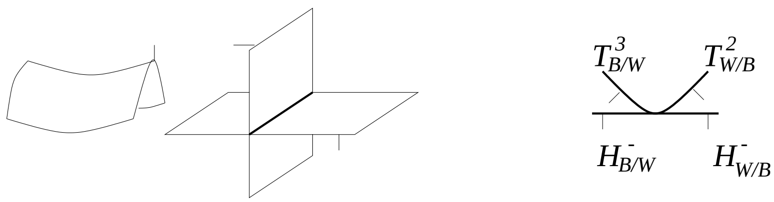

Figure 9: Bifurcation diagram for $(A_0^2|A_1^-)(A_0)|A_1$

See Figure 8. These can be formed from equations (5) and (6) to (13). The equations for the negative version of the singularity are

$$t^2_{B/W} + h^-_{B/W} - h^-_{W/B} - t^3_{W/B} = 0 \quad (23)$$
$$t^1_{B/W} + h^+_{B/W} - h^+_{W/B} - t^2_{W/B} = 0 \quad (24)$$
$$t^0_{B/W} + h^-_{B/W} - h^-_{W/B} - t^1_{W/B} = 0. \quad (25)$$

See Figure 9. We can use (5) to (7) and the equations (20) to (22) above to show they are trivial in our system.

The final tri-germ, the interaction of a cross cap and two transversely intersecting planes, gives six distinct pairs of black/white equations when we vary the orientations. Two of the pairs are given as (14) and (15). Another two are just the negatives of these two. The last two pairs are again negatives of each other. They can be described as

$$c^{+-}_{W/B} + t^2_{B/W} + c^{-+}_{B/W} + q^2_{W/B} - c^{+-}_{B/W} - t^1_{B/W} - c^{-+}_{W/B} = 0. \quad (26)$$

But these are trivial by (17) and (8) to (11).

Higher multi-germs. Before dealing with the quadri-germ let us note that from equation (14) we can deduce that

$$q^3_{B/W} = -q^3_{W/B}. \quad (27)$$

By (14) and (8) to (11) this equality is equivalent to $t^3_{B/W} - t^2_{B/W} - t^2_{W/B} + t^3_{W/B} = 0$. But this is true using (8) to (11), then (12) and (13), followed by (5) and (6).

The equations arising from the quadri-germ $(A_0^3|A_1)(A_0)$ are trivial. This is because the various orientations of the four surfaces lead to 8 distinct pairs of

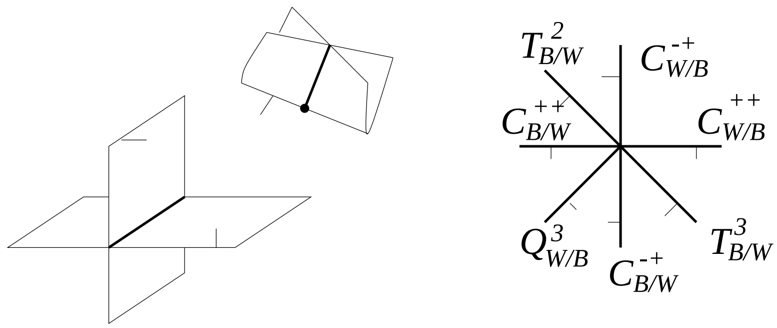

Figure 10: Bifurcation diagram for $A_0^2 S_0|A_1$ (following [1])

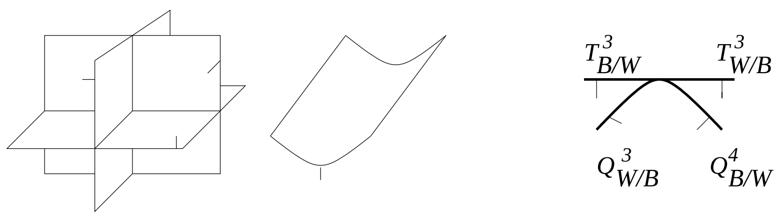

Figure 11: Bifurcation diagram for $(A_0^3|A_1)(A_0)$ (following [1])

equations.

$$t^2_{B/W} + q^2_{W/B} - q^3_{B/W} - t^2_{W/B} = 0 \quad (28)$$
$$t^1_{B/W} + q^2_{W/B} + q^3_{B/W} - t^1_{W/B} = 0 \quad (29)$$
$$t^1_{B/W} - q^3_{W/B} + q^2_{B/W} - t^1_{W/B} = 0 \quad (30)$$
$$t^2_{B/W} + q^3_{W/B} + q^2_{B/W} - t^2_{W/B} = 0 \quad (31)$$
$$t^3_{B/W} + q^3_{W/B} - q^4_{B/W} - t^3_{W/B} = 0 \quad (32)$$
$$t^0_{B/W} - q^3_{W/B} + q^4_{B/W} - t^0_{W/B} = 0 \quad (33)$$
$$t^0_{B/W} - q^4_{W/B} + q^3_{B/W} - t^0_{W/B} = 0 \quad (34)$$
$$t^3_{B/W} + q^4_{W/B} - q^3_{B/W} - t^3_{W/B} = 0. \quad (35)$$

See Figure 11. Using the $t^0_{W/B} - t^3_{B/W} = 0$ and $t^1_{W/B} - t^2_{B/W} = 0$ equations (given as (18) and (19) above), these 8 equations reduce to just two in the system. The equation arising from the last four above is trivial in the system using (8) to (15). Using (17) and $q^3_{B/W} = -q^3_{W/B}$ the equation arising from the other four equations reduce to $t^2_{B/W} - q^3_{B/W} - t^2_{W/B} = 0$. This equation is trivial using (14), (8), (9), (12), (13), (5) and (6).

The interaction of 5 planes, denoted A_0^5, also gives rise to trivial equations. The bifurcation diagram consists of 5 lines through the origin. The two parameter unfolding can be visualised by taking an ordinary triple point and then the fourth and fifth plane of the quintuple point are allowed to interact with the triple point.

Each line of the diagram consists of a Q^i stratum for some i. This is because firstly it is obvious only Q^i strata can occur. Secondly, given a point in the parameter space with a codimension 1 singularity, moving the plane not in the quadruple will not affect the type of singularity (except at the origin of course) or its coorientation.

Each Q^i line can be divided by removing the origin. The colours of the two resulting pieces will be different as they correspond to different positions of the fifth plane. That is, by drawing the fifth plane across a configuration we change the colours of the bubbles in the configuration.

The different equations can then be constructed by taking the diagrams arising from singularities near Q^4, Q^3, and Q^2. By passing the fifth plane through this configuration one gets four of the lines in the bifurcation diagram. The last line is the Q^i from which the configuration was constructed.

All the resulting equations are trivial in the system. This is because the strata cancel each other or in the case of Q^2 strata they are ignored because of equation (17). □

4.6 The Black and White Invariants

4.6.1 Integer invariants

Since there are 40 variables and 34 independent linear equations and up to an additive constant (we shall ignore additive constants in the theorems of this section) we get six solutions, which we list in the following theorem.

Theorem 4.3 *The space of black and white local invariants of generic mappings $f : N \to \mathbb{R}^3$ is six-dimensional and is generated by the following invariants.*

$$
\begin{aligned}
I_p^B &= 2(B_B + K_B) + C_B - C_W, \\
I_p^W &= 2(B_W + K_W) + C_W - C_B, \\
I_t^B &= 2(T_B^0 + T_B^2 + T_W^1 + T_W^3) + C_B^{+-} + C_B^{--} + C_W^{++} + C_W^{-+} \\
&\qquad +4(Q_W^4 - Q_B^4) + 2(Q_B^3 - Q_W^3), \\
&= 2(T_B^{\text{even}} + T_W^{\text{odd}}) + C_B^{*-} + C_W^{*+} + 4(Q_W^4 - Q_B^4) + 2(Q_B^3 - Q_W^3), \\
I_t^W &= 2(T_W^0 + T_W^2 + T_B^1 + T_B^3) + C_W^{+-} + C_W^{--} + C_B^{++} + C_B^{-+} \\
&\qquad +4(Q_B^4 - Q_W^4) + 2(Q_W^3 - Q_B^3), \\
&= 2(T_W^{\text{even}} + T_B^{\text{odd}}) + C_W^{*-} + C_B^{*+} + 4(Q_B^4 - Q_W^4) + 2(Q_W^3 - Q_B^3), \\
I_3^B &= E_B^2 - E_B^0 + H_B^- + T_W^0 + T_W^1 + T_B^2 + T_B^3 + C_B^{++} + C_B^{+-} + B_B^+ + K_B^+ \\
&\qquad +Q_B^3 - Q_W^3, \\
I_3^W &= E_W^2 - E_W^0 + H_W^- + T_B^0 + T_B^1 + T_W^2 + T_W^3 + C_W^{++} + C_W^{+-} + B_W^+ + K_W^+ \\
&\qquad +Q_W^3 - Q_B^3.
\end{aligned}
$$

Proof. One can just check that these invariants satisfy the jumps required for the system of equations in Lemma 4.2. □

Note the presence of the quadruple point strata, which is not observed in the ordinary Goryunov invariant case.

The invariants are naturally paired since if we change the colour 'at infinity', then we obtain an involution on the set of invariants:

$$
\begin{aligned}
I_p^B &\leftrightarrow I_p^W, \\
I_t^B &\leftrightarrow I_t^W, \\
I_3^B &\leftrightarrow I_3^W.
\end{aligned}
$$

By normalising on the standard embedding of N we see that I_p^B (resp. I_p^W) is the number of black (resp. white) cross caps, and that I_t^B (resp. I_t^W) is the number of black (resp. white) triple points. The invariants I_3^B and I_3^W are black and white versions of I_3 and can be set to be equal to zero on the standard embedding of N. One can also normalise in a similar way to that in [1] so that the invariants behave well under connected summation.

By adding $I_*^B + I_*^W$ we obtain Goryunov's invariants, where $* = p, t$ or 3:

$$\begin{aligned} I_p^B + I_p^W &= 2I_p, \\ I_t^B + I_t^W &= I_t, \\ I_3^B + I_3^W &= I_3. \end{aligned}$$

4.6.2 Integer invariants for immersions

In section 3 of [1] the case of maps that are immersions is treated. The resulting invariants are just restrictions of the local invariants. For the black and white case the same statement is true, restriction of the black and white integer invariants gives the complete set of local black and white invariants for immersions.

Theorem 4.4 *The space of black and white local invariants for immersions $f : N \to \mathbb{R}^3$ is four-dimensional and is generated by the following invariants.*

$$\begin{aligned} I_t^B &= T_B^0 + T_B^2 + T_W^1 + T_W^3 + 2(Q_W^4 - Q_B^4) + Q_B^3 - Q_W^3, \\ &= T_B^{\text{even}} + T_W^{\text{odd}} + 2(Q_W^4 - Q_B^4) + Q_B^3 - Q_W^3, \\ I_t^W &= T_W^0 + T_W^2 + T_B^1 + T_B^3 + 2(Q_B^4 - Q_W^4) + Q_W^3 - Q_B^3, \\ &= T_W^{\text{even}} + T_B^{\text{odd}} + 2(Q_B^4 - Q_W^4) + Q_W^3 - Q_B^3, \\ I_3^B &= E_B^2 - E_B^0 + H_B^- + T_W^0 + T_W^1 + T_B^2 + T_B^3 + Q_B^3 - Q_W^3, \\ I_3^W &= E_W^2 - E_W^0 + H_W^- + T_B^0 + T_B^1 + T_W^2 + T_W^3 + Q_W^3 - Q_B^3. \end{aligned}$$

Proof. These are found using the bifurcation diagrams for $A_0^2|A_2$, $A_0^3|A_2$, $(A_0^2|A_1^{\pm})(A_1)|A_1$, $A_0^3|A_2$, and A_0^5. The non-trivial equations can be taken to be (5), (6), (7), (16), (17), (18), (19), (20), (28), and (32). □

4.6.3 Mod 2 invariants

For mod 2 local invariants Goryunov found a further invariant: $I_4 = E^1 + H^+ + C^{+-} + C^{-+}$. This however does not split under consideration of black and white invariants. Equations (19) and (21) imply that $e_W^1 - e_B^1 = 0$ and, by (6) and (7), $h_B^+ = h_W^+$. Previously these equations were rendered trivial by $h_{B/W}^+ = 0$ but they are non-trivial in the mod 2 system.

We do however still have $q_{B/W}^2 = 0$ but this time this is deduced from equations (26) and (8) to (11).

So, from the 17 pairs of equations we eliminate (16) and (17) to work mod 2, we introduce (26) and one version of (21) will give $e_B^1 = e_W^1$. Thus we have 40 strata and 33 equations and so a 7-dimensional solution space.

Obviously reduction mod 2 of the black and white invariants will give mod 2 invariants. But this gives only 5 invariants as both I_p^B and I_p^W are the same. The two remaining invariants must come from Goryunov's original list and so in conclusion we have:

Theorem 4.5 *The space of mod 2 black and white local invariants for generic mappings $f : N \to \mathbb{R}^3$ is seven-dimensional and is generated by the following invariants.*

$$\begin{aligned}
I_p^B &= I_p^W = C_W + C_B, \\
I_p &= B + K \\
I_t^B &= C_B^{+-} + C_B^{--} + C_W^{++} + C_W^{-+} \\
I_t^W &= C_W^{+-} + C_W^{--} + C_B^{++} + C_B^{-+} \\
I_3^B &= E_B^2 - E_B^0 + H_B^- + T_W^0 + T_W^1 + T_B^2 + T_B^3 + C_B^{++} + C_B^{+-} + B_B^+ + K_B^+ \\
&\qquad + Q_B^3 - Q_W^3, \\
I_3^W &= E_W^2 - E_W^0 + H_W^- + T_B^0 + T_B^1 + T_W^2 + T_W^3 + C_W^{++} + C_W^{+-} + B_W^+ + K_W^+ \\
&\qquad + Q_W^3 - Q_B^3, \\
I_4 &= E^1 + H^+ + C^{+-} + C^{-+}.
\end{aligned}$$

4.6.4 Mod 2 invariants for immersions of spheres

Just as for the mod 2 invariants for generic maps we get mod 2 invariants for immersions of spheres. Again the black and white situation is not so straightforward.

We have the same equations as in the immersion case with the omission of the $q^2_{B/W} = 0$ and $h^+_{B/W} = 0$ and the addition of $e^1_B = e^1_W$ and $q^2_W = q^2_B$. The latter coming from the quintuple germ singularity. With 24 strata and 18 equations we deduce that the solution space is 6-dimensional; the solutions are the black and white immersion invariants mod 2 and Goryunov's invariants:

Theorem 4.6 *The space of mod 2 black and white local invariants for immersions from the 2-sphere to three-space is six-dimensional and is generated by the following invariants.*

$$\begin{aligned}
I_t^B &= T_B^0 + T_B^2 + T_W^1 + T_W^3 + Q_B^3 - Q_W^3, \\
&= T_B^{\text{even}} + T_W^{\text{odd}} + Q_B^3 - Q_W^3, \\
I_t^W &= T_W^0 + T_W^2 + T_B^1 + T_B^3 + Q_W^3 - Q_B^3, \\
&= T_W^{\text{even}} + T_B^{\text{odd}} + Q_W^3 - Q_B^3, \\
I_3^B &= E_B^2 - E_B^0 + H_B^- + T_W^0 + T_W^1 + T_B^2 + T_B^3 + Q_B^3 - Q_W^3, \\
I_3^W &= E_W^2 - E_W^0 + H_W^- + T_B^0 + T_B^1 + T_W^2 + T_W^3 + Q_W^3 - Q_B^3, \\
I_4 &= E^1 + H^+, \\
I_q &= Q.
\end{aligned}$$

5 Effectiveness of Black and White Invariants

To show the effectiveness of the black and white invariants we give an example of two distinct maps which cannot be distinguished by ordinary Goryunov invariants, but which have different black and white invariants.

Let $N = S^2$ the standard two-sphere and embed it in $\mathbb{R}^3$. Apply the changes E^0, K^+ and K^+ as shown in Figure 12. For this map we have $I_p = 2$, $I_t = 0$ and $I_3 = 1$.

Now develop two offshoots from the space as shown in Figure 13 and apply E^1 and E^0. For this map, denoted f, we have $I_p(f) = 2$, $I_t(f) = 0$ and $I_3(f) = 0$. Now we shall produce two maps by moving the offshoots, one movement crosses the C^{+-} stratum, the other crosses C^{++}. See Figure 14. For both maps we have $I_p = 2$, $I_t = 1$ and $I_3 = 1$ and so ordinary Goryunov invariants do not distinguish the two maps f_1 and f_2.

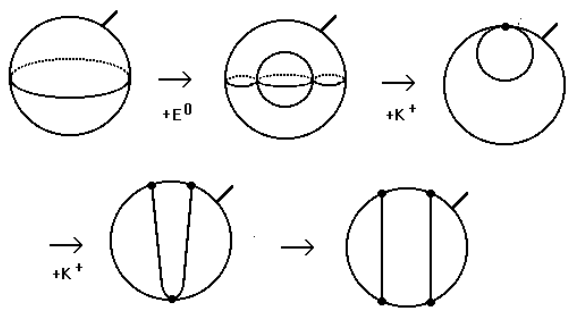

Figure 12: Creating three enclosed regions

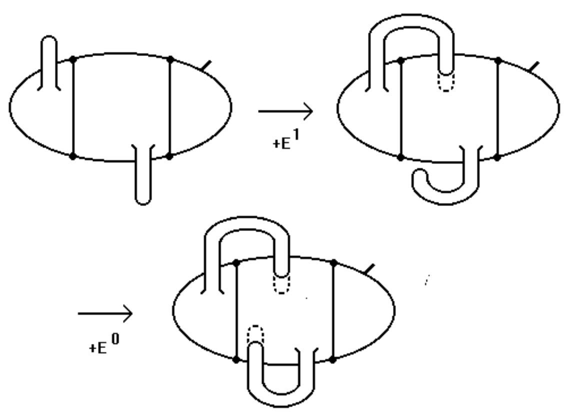

Figure 13: The image of f

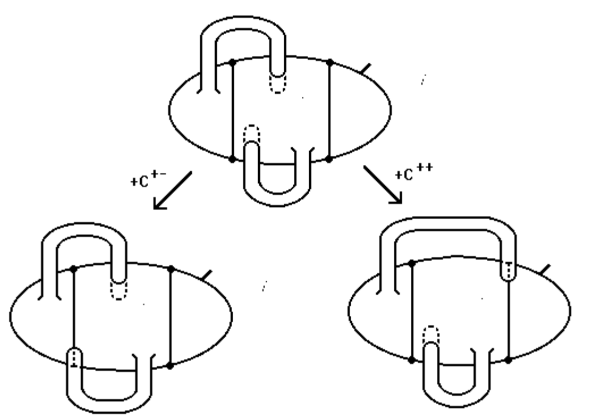

Figure 14: The image of f_1 and of f_2

Let us now colour the complements of the images with the chess board pattern with white the colour at infinity. Then to obtain the image of f we take the same route through Ω as before, this time noting the colouring:

$$E_B^0 \to K_B^+ \to K_B^+ \to E_W^1 \to E_W^0.$$

So for f we have

$$\begin{array}{rclrcl} I_p^B(f) & = & 4, & I_p^W(f) & = & 0, \\ I_t^B(f) & = & 0, & I_t^W(f) & = & 0, \\ I_3^B(f) & = & 1, & I_3^W(f) & = & -1. \end{array}$$

To get f_1 we go via C_W^{+-}. Thus

$$\begin{array}{rclrcl} I_p^B(f_1) & = & 3, & I_p^W(f_1) & = & 1, \\ I_t^B(f_1) & = & 0, & I_t^W(f_1) & = & 1, \\ I_3^B(f_1) & = & 1, & I_3^W(f_1) & = & 0. \end{array}$$

To get f_2 we go via C_W^{++}. Thus

$$\begin{array}{rclrcl} I_p^B(f_2) & = & 3, & I_p^W(f_2) & = & 1, \\ I_t^B(f_2) & = & 1, & I_t^W(f_2) & = & 0, \\ I_3^B(f_2) & = & 1, & I_3^W(f_2) & = & 0. \end{array}$$

We conclude that f_1 and f_2 are not isotopic as $I_t^*(f_1) \neq I_t^*(f_2)$, where $* = B$ or W. That is, the triple points are different colours.

6 Concluding Remarks

6.1 Integral invariant

A Rokhlin type invariant $I_\int$ is defined in [1] using a degree function and the formula

$$I_\int = \int_{\mathbb{R}^3 \setminus im(f)} deg(u) d\chi(u) - \sum_t deg(t) - \frac{1}{2} \sum_p deg(p).$$

It is tempting to generalise this by defining a black invariant version to be the sum over only black regions, triples and pinch points. However this is not a local invariant. The increase in $I_\int^B$ for C_B^{++} does not depend on the stratum but on the degree function:

$$\Delta(I_\int^B) = \{d + (d+2) + 2 - \frac{1}{2}(d+1)\} - \{d + d\} = \frac{1}{2}(d-3).$$

Thus it cannot be a local invariant and there does not appear to be an obvious way to define black and white Rokhlin type invariants so that their calculation is by black and white local invariants.

6.2 Wavefronts

With a complete classification of codimension 2 wavefronts in three-space one could find the complete set of order one local invariants for wavefronts. These should include a Legendrian type invariant (i.e. similar to Goryunov's Rokhlin type invariant and not invariants of Legendrian embeddings up to isotopy) and the numbers of swallowtails, triples points and the crossing of cuspidal edges and planes. This list may even be exhaustive. Whatever the outcome we should be able to define black and white invariants in this case as well.

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