Topology of Differentiable Mappings

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Abstract

To study the topology of a differentiable mapping one can consider its image or its fibres. A proportion of this survey paper looks at how the latter can be studied in the case of singular complex analytic maps. An important aspect of this is study of the local case, the primary object of interest of which is the Milnor Fibre. More generally, Stratified Morse Theory is used to investigate the topology of singular spaces. In the complex case we can use rectified homotopical depth to generalize the Lefschetz Hyperplane Theorem. In the less studied case of images of maps we describe a powerful spectral sequence that can be used to investigate the homology of the image of a finite and proper map using the alternating homology of the multiple point spaces of the map.

Contents

1 Introduction

Given a smooth map $f : N \to P$ from one manifold to another, it is natural to ask for a description of the topology of $f(N)$ and $f^{-1}(p)$ for $p \in P$. One can see that for the topology of a map, generally, we have this option of studying *images* or *preimages*.

The theory for the latter is considerably more advanced than the former since Sard's Theorem and the Implicit Function Theorem tell us that, usually, the preimage of a smooth map is a manifold, while experience shows that, in general, the image of a map is singular. The selection of topics in this survey reveals that this is the case and attempts to redress the balance by presenting some lesser known results and, more importantly, techniques, in the study of the topology of images.

The dichotomy between images and preimages is presented in Section 2. Some fundamental examples such as the simple singularities of functions and the Whitney umbrella/cross-cap are presented.

The basic building block of the topology of differentiable maps is the Milnor fibre. This is presented in Section 3. This describes the local change, i.e., within a sufficiently small ball, of the topology of the singularity as one moves from the critical point to a nearby non-critical point. That is, from the singular fibre to a nearby non-singular one. Since the Milnor fibre is, in general, a non-trivial fibration over a circle we have monodromy. This is described in Section 4

In his 1978 obituary of Morse in [60], Smale said 'Morse theory is the single greatest contribution of American mathematics' (perhaps not surprisingly as the result formed the backbone of Smale's own Fields-medal-winning work on the h-cobordism theory and the higher dimensional Poincaré Conjecture. However, this obituary did cause discussion due to its critical nature). Morse Theory is well-known and covered in the highly readable [48] so it is the generalization to singular spaces, in fact, more properly, stratified spaces by Goresky and Macpherson that we describe Section 6. Their original intention had been to generalize the Lefschetz Hyperplane Theorem to the case of their (then) recently invented Intersection Cohomology. In pursuing this they invented Stratified Morse Theory. This subject is now fairly advanced, one can see by looking through [57] at the level of sophistication now possible. However, this sophistication is underused and there are many subjects to which it could be applied that have yet to be explored. The stratification of spaces is detailed in Section 5 and Stratified Morse Theory in Section 6.

The main result of Morse theory is that one can build up the topology of a manifold by placing on it a generic function that has non-degenerate singularities. The topology can then be described by attaching cells of dimension that depend on the index of the second differential of the the critical points of the map. In the stratified case we have to calculate the Morse index at a point but also have to take into account how the function behaves with respect to the space transverse to the stratum containing the point. The local intersection of the singular space and a manifold transverse to the stratum is called the normal slice. To apply Stratified Morse Theory we need to be able to describe the topology of the Morse function on this space. Unlike the usual Morse index, no very simple number exists to measure this topology. One method is to use Rectified Homotopical Depth, a concept, introduced by Grothendieck, which is analogous to the idea of depth from commutative algebra. In that theory regular rings and complete intersection rings have maximal depth, i.e., equal to the ring's dimension, we have that manifolds and local complete intersections have maximal Rectified Homotopical Depth, i.e., equal to the complex dimension of the space. Section 7 shows how this notion can be used with Stratified Morse Theory to describe the topology of certain complex analytic varieties in $\mathbb{CP}^n$.

Another interesting and greatly underutilized generalization of Stratified Morse Theory is given in Section 8. This is a relative version, i.e., for a stratified map $f : X \to Y$ between Whitney stratified sets a stratified Morse function on Y is used to describe the topology of X.

In the last sections we see a spectral sequence that allows us to deal with the topology of images. The potential applications of this sequence are quite large.

First, Section 9 gives examples of how images behave and discusses the multiple point spaces for a map. For a continuous map $f : X \to Y$ the kth multiple point space is

$$D^k(f) = \text{closure}\{(x_1, \dots, x_k) \in X^k \mid f(x_1) = \cdots = f(x_k), \text{ for } x_i \neq x_j, i \neq j\}.$$

The key here is that the image is very hard to describe, for example, the image of a real polynomial map may not be a real algebraic set. Now $D^k(f)$ can often be described as the zero-set of map and hence we can use the theory developed for level sets to produce a theory for images. Furthermore, $D^k(f)$ has a lot of symmetry since S_k, the group of permutations on k objects, acts on it, and this is exploited in describing the topology of the image. In fact, we need to focus on the the alternating homology of $D^k(f)$, that is, chains on $D^k(f)$ that are anti-invariant under the action of S_k. This alternating homology then forms the E^1 terms of a spectral sequence for a wide class of finite and proper maps. This sequence is called the Image Computing Spectral and is described, with examples, in Section 10.

2 Manifolds and singularities

Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with $n \geq 1$. A fundamental question in mathematics is 'What is the topology of the level sets $f^{-1}(c)$ for $c \in \mathbb{R}$?' Whitney showed the following.

Theorem 2.1 (Whitney [7]) *Let X be a closed set in $\mathbb{R}^n$. Then there exists a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f^{-1}(0) = X$.*

Due to the wildness of closed sets – consider the pathological examples of the Cantor set and Hawaiian earrings – this, of course, means that finding a general structure theorem on the level set of an arbitrary smooth function f is essentially hopeless – and we have not even considered maps into higher dimensional spaces yet.

So we begin by specializing and look at a fundamental structure theorem for the level sets of certain smooth maps. First some definitions. Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a smooth map where n and p are arbitrary. If the differential at a point $x \in \mathbb{R}^n$ is surjective, then we say that f is a *submersion* at x and say that x is a *regular point* of f. If all points of $f^{-1}(c)$ are regular points, then we say that that c is a *regular value* of f. (This includes the case that c is not a value of f!)

Then we have the following.

Theorem 2.2 *Suppose that c is a regular value of f. Then $f^{-1}(c)$ is a $(n-p)$-dimensional submanifold of $\mathbb{R}^n$.*

The first remedy for the problem posed by Whitney's theorem is Sard's theorem which says that, in general, the level set is a manifold.

Theorem 2.3 (Sard's Theorem [56]) *The set of non-regular values of f has Lebesgue measure zero.*

Thus for mappings we have that, in general, a fibre is a manifold. For images the situation is not so good. However, we do have that if we have a map $f : \mathbb{R}^n \to \mathbb{R}^p$ with $n < p$ and maximal possible rank (i.e., n) at the point $x \in N$, then there is a neighbourhood U of x that maps into P such that $f(U)$ is a submanifold of P.

These last two theorems can be proved from the Inverse Function Theorem or Implicit Function Theorem. (These two theorems are in fact equivalent as each can be proved from the other.)

Theorem 2.4 (Implicit Function Theorem) *Suppose that $f : \mathbb{R}^{n+r} \to \mathbb{R}^r$ is a smooth map defined on a neighbourhood of $(x_0, y_0) \in \mathbb{R}^n \times \mathbb{R}^r$ with $f(x_0, y_0) = c$. If the $r \times r$ matrix*

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial y_1} & \cdots & \dfrac{\partial f_1}{\partial y_r} \\ \vdots & & \vdots \\ \dfrac{\partial f_r}{\partial y_1} & \cdots & \dfrac{\partial f_r}{\partial y_r} \end{pmatrix}$$

is non-singular at (x_0, y_0), then there exists a neighbourhood of U of x_0 in $\mathbb{R}^n$ and V of y_0 in $\mathbb{R}^q$ such that for all x in U there is unique point $g(x)$ in V with $f(x, g(x)) = c$. Furthermore, g is smooth.

The preceding can be generalized using the notion of transversality (which we shall use later in a different context).

Definition 2.5 *Let U and V be submanifolds in $\mathbb{R}^n$. Then, U and V are* transverse at the point $x \in U \cap V$ *if*

$$T_x U + T_x V = \mathbb{R}^n.$$

That is, the sum of the tangent spaces of U and V gives the tangent space to $\mathbb{R}^n$. We say that U and V are transverse *if they are transverse for all points in $U \cap V$.*

If U and V do not intersect, then automatically we say that the spaces are transverse.

The notion of transversality is very important as one would expect that two randomly chosen submanifolds would be transverse. In fact, if they were not, then using Sard's theorem one could perturb them slightly so that we had transversal intersection. Thus transversality is in some sense 'generic'.

We can produce a relative version of transversality.

Definition 2.6 *Let $f : N \to P$ be a smooth map between the manifolds N and P. Suppose that C is a smooth submanifold of P. Then, f is said to be transverse to C at the point $x \in N$ if either $f(x) \notin C$ or the image of the tangent space to N under the differential $d_x f$ is transverse to the tangent space of C at $f(x)$ in P. That is,*

$$d_x f(T_x N) + T_{f(x)} C = T_{f(a)} P.$$

We say that f is transverse to C *if it is transverse to C at all points $x \in N$.*

Then we can generalize the structure theorem for level sets.

Theorem 2.7 *Suppose that $f : N \to P$ is a smooth between manifolds with C a submanifold of P. If f is transverse to C, then $f^{-1}(C)$ is a submanifold of N of codimension equal to the codimension of C in P.*

Singularities of spaces and mappings

Let us look at the case of $f : \mathbb{R}^n \to \mathbb{R}^p$, where $n > 0$ and $p > 0$. Loosely speaking, we know that if the differential of f has maximal rank at a point then we have a submersion or immersion; in the former the preimage of the value is a manifold and in the latter the image is a submanifold. Thus let us turn our attention to the case where the map does not have maximal rank.

Definition 2.8 *We define a* singular point of f *to be a point x where $d_x f$ has less than maximal rank.*

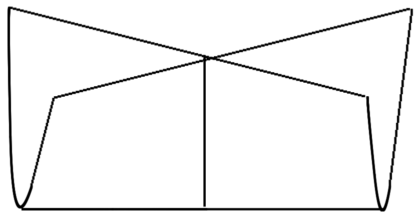

Figure 1: The Whitney Cross-Cap.

Example 2.9 *(i). Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_n^2$. Then f is a* Morse singularity*. It is well-known that Morse proved that a function with a critical point such that the second differential is non-degenerate (equivalently, the square matrix of second derivatives is non-singular) is equivalent to such a Morse singularity (up to addition of a constant), see [48]. He also proved that such singularities are dense and stable – i.e., 'most' maps are Morse and they cannot be removed by perturbation.*

This classification has probably the profoundest effect in the theory of topology of manifolds as it leads to Morse Theory, of which more will be said later.

(ii). Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $f(x, y) = (x, xy, y^2)$. This (and its image) is known as the Whitney cross-cap *or* Whitney umbrella*. See Figure 1. This is also called the Whitney-Cayley cross-cap since it was known to Cayley. See [3] p217 for references and a discussion of his work in this area.*

The singular set is the origin in $\mathbb{R}^2$. The image has singularities – points where the set is non-manifold – along the Z-axis in $\mathbb{R}^3$, and, apart from the origin, these are transverse crossing of manifolds.

(iii). The image of the Whitney cross cap is of codimension 1 in the codomain. If we attempt to find a polynomial that defines the image of f as a hypersurface we can try $h : \mathbb{R}^3 \to \mathbb{R}$ given by $h(X, Y, Z) = Y^2 - X^2 Z$. However, the image of f is actually a semi-algebraic set and so the zero-set of this h gives the image of f with a 'handle' that consists of the Z-axis. With this added to the image in Figure 1 one can see why the map is referred to as an umbrella.

If we work over the complex numbers rather than the reals, then the image and the zero-set descriptions coincide.

The last example shows the wide variety of behaviour that can occur for the image of a fairly simple algebraic map. One gets even stranger examples of if one considers images of smooth maps. For example, we can show that the corner of a cube can be produced from the image of an infinitely differentiable map.

Example 2.10 *Let $\phi : \mathbb{R}^2 \to \mathbb{R}^+$ be the smooth map given by*

$$\phi(x, y) = \begin{cases} e^{-1/x^2} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}$$

Let r_j be the map that rotates the plane about the origin through $\pi j/3$ radians. Let $\psi_1 = \phi \circ r_0 + \phi \circ r_1$, $\psi_2 = \phi \circ r_2 + \phi \circ r_3$ and $\psi_3 = \phi \circ r_4 + \phi \circ r_5$.

Then each ψ_i is a smooth function on the plane that is zero in a region bounded by two rays from the origin that are $2\pi/3$ radians apart. The interiors of these three regions do not overlap; the overlaps of the closures correspond to edge points of the corner of the cube.

The image of the map $h = (\psi_1, \psi_2, \psi_3)$ gives the corner of a cube.

Let us look at some more interesting examples, this time of less pathological singularities.

Example 2.11 *(i). The simple singularities, A_k, D_k and E_k of maps $f : \mathbb{R}^2 \to \mathbb{R}$ are defined to be*

$$\begin{array}{ll} A_k : & x^{k+1} + y^2, \text{ for } k \geq 1, \\ D_k : & x^2 y + y^{k-1}, \text{ for } k \geq 4, \\ E_6 : & x^3 + y^4, \\ E_7 : & x^3 + xy^3, \\ E_8 : & x^3 + y^5. \end{array}$$

The relation between the notation and that of simple Lie groups is not coincidental. See, for example, [2] page 99.

(ii). The Whitney cusp is given by $f : \mathbb{R}^2 \to \mathbb{R}^2$ is important in the study of maps between surfaces. It is given by $f(x, y) = (x, y^3 + xy)$. It is stable in the sense that if we perturb the map slightly, then there is some change of coordinates in the source and target that maps the perturbation back to the Whitney cusp.

The word 'cusp' is used in name of the last example because the *discriminant* of the map is a cusp.

Definition 2.12 *Let $f : \mathbb{R}^n \to \mathbb{R}^p$ be a smooth with n and p arbitrary. Let C be the critical set of f, i.e., the points of $x \in \mathbb{R}^n$ such that* $\operatorname{rank} d_x f < p$.

Then the discriminant *of f is $f(C)$, the image of the critical set.*

Example 2.13 *For the Whitney cusp the discriminant is diffeomorphic to the standard cusp, i.e., the image of $t \mapsto (t^2, t^3)$, and hence the name for the Whitney cusp.*

The discriminant of the map is a very powerful invariant. It is a space which contains a lot of information concerning the map. In many cases given the discriminant it is possible to recover the map, see for example, [8] and [11].

3 Milnor Fibre

We shall discuss later how classical Morse theory can be used to great effect in describing the topology of manifolds. For the moment we shall note that the essence of the theory is that the local behaviour of a function, in particular its singularities, is used to describe the global behaviour of the topology. In this section we consider the case of complex singularities, a field pioneered by the ancients (i.e., mathematicians pre-1900), but the approach is more recent and follows on from the seminal work of Milnor [49]. Whilst Milnor's initial contribution cannot be underestimated it should be noted that many mathematicians have contributed to the theory – too many to do justice to in this paper. However, special mention should be made Lê Dũng Tráng, he has perhaps done more than any other to advance and popularize the theory of the Milnor fibre.

First, it should be noted that, in contrast to the case of real functions, for complex functions there is no local change in topology as one passes through a critical value. Instead one has to look at monodromy which is tackled in the next section. In this section we look at the local description of the fibres near to the singular fibre. The Milnor fibre is a fibre nearby to a singular fibre and is considered to be a local object, that is, we intersect with a neighbourhood.

The Milnor fibre of a complex function on a manifold

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a non-constant complex analytic map such that $f(0) = 0$ and $n \geq 1$. We shall be interested in $f^{-1}(0)$, particularly at the origin in $\mathbb{C}^{n+1}$, and $f^{-1}(t)$ where $t > 0$ is small and $f^{-1}(t)$ is non-singular.

We restrict ourselves to local behaviour. Let us fix our notation for this. The sphere of radius ϵ centered at 0 in $\mathbb{C}^{n+1}$ is denoted S_ϵ; it bounds the closed ball B_ϵ; the open ball will be denoted B_ϵ°.

Our first result is the following:

Theorem 3.1 (Conic structure theorem) *There exists $\epsilon_0 > 0$ such that*

(i). $S_\epsilon \cap f^{-1}(0)$ *is homeomorphic to* $S_{\epsilon_0} \cap f^{-1}(0)$ *for all* $0 < \epsilon \leq \epsilon_0$*;*

(ii). Cone $\left(S_{\epsilon_0} \cap f^{-1}(0)\right)$ *is homeomorphic to* $B_{\epsilon_0} \cap f^{-1}(0)$*.*

This Conic Structure Theorem holds for a far wider class of objects than just complex analytic sets, for example, Whitney stratified sets, a class we shall define later.

Definition 3.2 *The space* $L = S_\epsilon \cap f^{-1}(0)$ *is called the* real link of f at 0.

In his original ground-breaking text [49] Milnor showed this $(2n-1)$-dimensional space is $(n-2)$-connected. That is, $\pi_i(L) = 0$ for $0 \leq i \leq n-2$. (By convention, π_0 is trivial if and only if the space is path-connected). This was later improved by Hamm to complete intersections as discussed below.

We f has an *isolated singularity at* 0 if there exists an open neighbourhood U of 0 such that $U \cap f^{-1}(0) \backslash \{0\}$ is a manifold. Note that this includes the case that f is in fact non-singular at 0 – a standard, if perverse, use of terminology.

In the case that f has an isolated singularity, then K is a manifold. This has many interesting interpretations. For example, if $n = 1$, then the level set of f is a complex curve and so K is a knot (in fact a link, hence the name) in the 3-manifold S_ϵ (and hence in $\mathbb{R}^3$ as K does not fill the three manifold). Results relating these knots to analytic curves and vice versa were given in [49]. A short survey of more recent results can be found in [67].

If one goes to higher dimensions, then one can produce exotic spheres. Kervaire and Milnor showed in [34] that there exists manifolds homeomorphic to spheres which are not diffeomorphic to the standard differentiable structure on the sphere. These are called *exotic spheres*. Brieskorn gave the following example in [6].

Example 3.3 *Let* $f : \mathbb{C}^5 \to \mathbb{C}$ *be given by*

$$f(x, y, z, t, u) = x^2 + y^2 + z^2 + t^3 + u^{6k-1}.$$

Then, for $1 \leq k \leq 28$*, the link of the origin of* $f^{-1}(0)$ *is a topological* 7*-sphere. Furthermore, these give the* 28 *different types of exotic* 7*-spheres.*

The proof of this involves Smale's proof of the higher dimensional Poincaré conjecture and analysis of the monodromy of the singularity.

Thus, the link of a singularity is an interesting space in its own right. There is much to be investigated about it, particularly for surfaces and for non-isolated singularities.

We now turn to the Milnor fibre of a singularity. One of the key results of [49] is the existence of a fibration connected with a neighbourhood of the singularity. Milnor originally defined this in a different fashion to the standard one about to be given, which is due to Lê in [38].

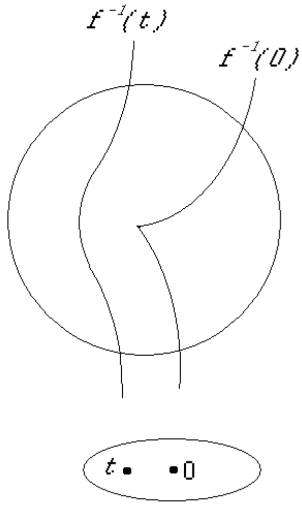

Figure 2: Schematic diagram of Milnor fibration

Theorem 3.4 (Milnor Fibration Theorem) *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a complex analytic map and ϵ is taken small enough so that S_ϵ defines the real link of f. Let D^*_δ be the set $\{t \in \mathbb{C} : 0 < |t| < \delta\}$.*

Then, $f : B^\circ_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$ is a smooth locally trivial fibration for $0 < \delta << \epsilon$.

In fact, $f : B_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$ is locally trivial topological fibration with $f : S_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$ a subfibration.

Furthermore, if f has an isolated singularity, then this subfibration extends over $D_\delta = \{t \in \mathbb{C} : |t| < \delta\}$.

Unsurprisingly, the proof of the first part of this involves the Ehresmann Fibration Theorem. The second can be proved using the First Thom–Mather Isotopy Lemma which will be discussed later.

Definition 3.5 *The fibre of $f : B^\circ_\epsilon \cap f^{-1}(D^*_\delta) \to D^*_\delta$ is called the* Milnor fibre of f *and is denoted by F°_f, or F° if no confusion will result. The closed fibre is F_f or F. The boundary of F is denoted ∂F and, as can be seen from the theorem, this is also the fibre of a fibration.*

A schematic picture of the fibrations is given in Figure 2. It should be noted that because we can collar ∂F in F that F° and F are homotopically equivalent and are effectively interchangeable in many theorems.

Example 3.6 *(i). Recall that $f(x, y, z) = y^2 - x^2 z$ defines the Whitney Umbrella of Example 2.9(ii). The Milnor fibre is homotopically equivalent to a sphere, see Example 3.16.*

(ii). Let $f(x_1, x_2, \ldots, x_n, x_{n-1}) = x_1 x_2 \ldots x_{n+1}$. Then, it is easy to calculate that the Milnor fibre of f is homotopically equivalent to $(S^1)^{n+1}$.

One of the reasons that the Milnor fibre is such a useful construction is it is a topological invariant in the following sense.

Definition 3.7 *Two functions f and g from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ with $f(0) = g(0) = 0$ are* topologically equivalent at 0 *if there exists a homeomorphism $h : U \to V$, where $U, V \subseteq \mathbb{C}^{n+1}$ are open sets containing the origin, such that $f = g \circ h$.*

We have the following crucial theorem.

Theorem 3.8 ([36]) *Suppose that f and g are topologically equivalent. Then, their Milnor fibres are homotopically equivalent.*

Since the Milnor fibre is a Stein space, by Hamm [22] (see [17] for a simpler proof), it has the homotopy type of a CW-complex of real dimension equal to its complex dimension, i.e., n. This dimension is called *the middle dimension.* Thus we can place an upper bound on the non-vanishing of homology. Also, we can place a lower bound on the non-vanishing of reduced homology groups of the Milnor fibre, and in fact can do this for homotopy groups.

Proposition 3.9 (Kato-Matsumoto [33]) *If the singular set of f at 0 has dimension s, then F_f is $(n-s-1)$-connected.*

From Example 3.6(ii) we see that this bound is in some sense sharp. However, the converse of the theorem is not true and so one would like a more accurate statement. This has proven difficult to find.

Also, we do not have many general theorems describing the homology groups between $n-s-2$ and n, and so this is an area requiring more research. However, one example of such a theorem is given by Némethi [51], where some seriously heavy topological work is employed to the case of compositions of functions, that is to maps $f : \mathbb{C}^{n+1} \to \mathbb{C}^2$ defining complete intersections with an isolated singularity and curve singularities $g : \mathbb{C}^2 \to \mathbb{C}$. Let us define these.

Example 3.10 *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}^2$ be such that $f^{-1}(0)$ is of dimension $n-1$ and has an isolated singularity at the origin in $\mathbb{C}^{n+1}$. Suppose that $g : \mathbb{C}^2 \to \mathbb{C}$ defines a reduced curve singularity. Then $h = g \circ f$ is called a* Generalized Zariski singularity*.*

In this case, the singular set of h coincides with the fibre of f and so has codimension 1 in the fibre of h. Then, $H_*(F, \mathbb{Z}) = 0$ for $* \neq 0,\ 1,\ n$. See [51], [52] and, for more recent work, [25].

Isolated singularities and the unreasonable effectiveness of the Milnor number

An important corollary of Proposition 3.9 was first proved by Milnor.

Corollary 3.11 (Milnor, [49]) *If f has an isolated singularity at 0, then F_f has the homotopy type of wedge of spheres of dimension n.*

Recall that the wedge of two topological spaces is their one point union. Hence, a *wedge of spheres* (also known as a *bouquet of spheres*) is a collection of spheres, each member of which has a special point identified to the special point on the other members. An interesting feature of investigations of the local behaviour of Milnor fibres is the appearance of many results involving the wedge of spaces. This will be seen more clearly in Theorem 5.11.

Definition 3.12 *The number of the spheres in the wedge is called the* Milnor number *of f at 0, and is denoted $\mu(f)$.*

The Milnor number is a surprisingly effective topological invariant. Probably, one reason for this is that it can be calculated algebraically with ease: Suppose that coordinates on $\mathbb{C}^{n+1}$ are given by $x_1, x_2, \dots, x_{n+1}$ and that $\mathbb{C}\{x_1, x_2, \dots, x_{n+1}\}$ denotes the ring of convergent power series at 0. Then, Milnor [49] showed that

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, x_2, \dots, x_{n+1}\}}{\left\langle \dfrac{\partial f}{\partial x_1}, \dots, \dfrac{\partial f}{\partial x_{n+1}} \right\rangle}.$$

The ideal in the denominator is called the *Jacobian ideal of* f and the quotient ring is called the *Milnor algebra of* f.

Example 3.13 *Consider the case of a Morse singularity, i.e.,*

$$f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2 + c.$$

In this case the Milnor number is easily seen to be equal to 1.

Since we can produce knots via the real link of a singularity this means that we can use μ as an invariant of knots constructed in this way.

Furthermore, $\mu(f)$ is related to the unfolding properties of f. On the space of complex analytic functions we can place a natural equivalence relation.

Definition 3.14 *Two functions* f *and* g *are* right-equivalent *if there exists a biholomorphism* $h : U \to V$ *of open sets* U *and* V *in* $\mathbb{C}^{n+1}$ *such that* $f = g \circ h$.

The orbit of a function can be referred to as a singularity type since all functions in the orbit have the same type of singularity.

The codimension of the orbit in the space of all complex analytic functions is $\mu(f)$. By taking a set of functions $\{\alpha_i(x)\}_{i=1}^{\mu(f)}$ that projects to a basis of the $\mathbb{C}$-vector space of the Milnor algebra of f we can produce an *unfolding* of f:

$$F(x, \lambda) = f(x) + \sum_{i=1}^{\mu(f)} \lambda_i \alpha_i(x).$$

The idea of this concept is that (up to isomorphism) this family of functions contains all functions near to f.

Thus, it can be seen that μ is linked closely to the topology of the Milnor fibre and that it measures how complicated the singularity is by measuring how 'deep' within the space of functions the singularity sits.

New Milnor fibres from old

As usual in mathematics one would like to construct new examples of objects from old examples in such a way that the properties of the new can be calculated from that of the old.

In this vein is the very useful following theorem first investigated by Sebastiani and Thom and proved in more generality by Sakamoto in [55]. First recall that the *join* of two topological spaces X and Y is denoted by $X * Y$ and is defined as $X \times [0, 1] \times Y$ with the following identifications:

(i). $(x, 0, y) \sim (x', 0, y)$ for all $x, x' \in X$ and $y \in Y$,

(ii). $(x, 1, y) \sim (x, 1, y')$ for all $x \in X$ and $y, y' \in Y$.

Proposition 3.15 (Sebastiani-Thom Theorem, [58], [55]) *Let* $f : \mathbb{C}^r \to \mathbb{C}$ *and* $g : \mathbb{C}^s \to \mathbb{C}$ *be complex analytic maps with* $f(0) = g(0) = 0$. *Define*

$$f \oplus g : \mathbb{C}^{r+s} \to \mathbb{C} \text{ by } (f \oplus g)(x, y) = f(x) + g(y).$$

Then,

$$F_{f+g} \text{ is homotopically equivalent to } F_f * F_g.$$

Consequently, $\mu(f \oplus g) = \mu(f)\mu(g)$.

There are many generalizations of this theorem to different settings. For example, [45] shows how to generalise the underlying isomorphism to one in the derived category for very general singular functions. This paper also includes a number of references to other results in the area.

The concept of this result has an interesting application in Stratified Morse Theory when considering the splitting of local Morse data into local Normal and Tangential Morse Data, see Section 6.

Example 3.16 *In the complex setting the Whitney umbrella is the image of* $f : \mathbb{C}^2 \to \mathbb{C}^3$ *be given by* $f(x, y) = (x, xy, y^2)$. *This set is also given as a zero-set using* $h : \mathbb{C}^3 \to \mathbb{C}$ *defined by* $h(X, Y, Z) = Y^2 - X^2 Z$. *From the Sebastiani-Thom result we can see that the Milnor fibre of* h *is the suspension of the Milnor fibre of* $g(X, Z) = X^2 Z$. *Since,* g *is homogeneous, its Milnor fibre is given by the solution set of* $X^2 Z = 1$ *and so can be given by*

$$\left(X, \frac{1}{X^2} \right) \text{ for } X \neq 0.$$

The fibre is thus homeomorphic to $\mathbb{C}\backslash\{0\}$ *which is in turn homotopically equivalent to a circle. Therefore, the Milnor fibre of the Whitney umbrella is homotopically equivalent to the suspension of a circle, i.e., a* 2*-sphere.*

Complete intersections

Much of the preceding theory was generalized from functions to maps. Let $f : \mathbb{C}^{n+r} \to \mathbb{C}^r$ be a complex analytic map such that $f(0) = 0$ and, for simplicity, assume that $n, r \geq 1$. Again, we can define the link of $f^{-1}(0)$ as $S_\epsilon(0) \cap f^{-1}(0)$ for small enough ϵ. Hamm's theorem in [21] states that it is at least $(n-2)$-connected.

However, as we have no guarantee that a non-constant map will give a fibre of dimension n, we restrict to the case that $f^{-1}(0)$ has dimension n (the dimension we would expect in generic situations) and that it has an isolated singularity at the origin. In this case we say that f defines an *isolated complete intersection singularity* which is traditionally abbreviated to ICIS.

Again we find a Milnor fibration over the non-critical values in the target. As we assume that the singularity is isolated, we have a bouquet theorem: The Milnor fibre of an ICIS is a bouquet of n-spheres. It is quite common to find these bouquet theorems, we will see a reason for this in Theorem 5.11.

The number of spheres is again called the *Milnor number*. This is harder to calculate than the function case as it cannot be described as the quotient of some relatively simple ideal (or module); it can however be given be calculated quite effectively in low codimension by an alternating sum of numbers, see [37] or [41] page 76-77. The latter book is the standard reference for isolated complete intersection singularities.

For an ICIS one can define the *Tjurina number* as the dimension of the space

$$\tau(f) = \dim_{\mathbb{C}} \frac{(\mathbb{C}\{x_1, x_2, \dots, x_{n+r}\})^r}{\{\langle f_1, f_2, \dots, f_r \rangle e_i\}_{i=1}^r + \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n+1}} \right\rangle}$$

where e_i is the standard basis, i.e., a column vector with a 1 in position i and 0 elsewhere. A basis of the space used in the definition of τ can be used to construct an unfolding of f that has all nearby functions up to $\mathcal{K}$-equivalence. See [66].

In the case of function $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with an isolated singularity we see that

$$\tau(f) = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x_1, x_2, \dots, x_{n+1}\}}{\left\langle f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_{n+1}} \right\rangle}.$$

It is easy to see that in this case that $\tau(f) \leq \mu(f)$ with equality if f is quasi-homogeneous.

Theorem 3.17 *For an ICIS we have $\tau \leq \mu$ with equality if f is quasi-homogeneous.*

The proof of this is considerably harder than the function version and relies on Hodge theory, see [42]. In one respect it is unsatisfactory in that it does not provide much insight into why the result is true, however, in 20 years no-one has improved upon the proof.

4 Monodromy

The Milnor fibration is a fibration over a punctured disc and so for a moment let us consider it as a fibration over the circle S^1. As this fibration is locally trivial we can consider what happens to a point $x \in f^{-1}(t)$ for any $t \in S^1$ as we go round a loop in S^1. Executing a complete loop gives a homeomorphism from F to F, which, in general, is not the identity.

Definition 4.1 *The homeomorphism $h : F \to F$ is the* geometric monodromy *of f. The map induced on homology*

$$h_* : H_*(F;\mathbb{Z}) \to H_*(F;\mathbb{Z})$$

is called the (classical) monodromy operator *of f.*

We can prove the existence of this map by taking a vector field on S^1 and using the Ehresmann Fibration Theorem to produce a corresponding vector field on the total fibre space. The geometric monodromy is then given by integrating this vector field.

Now consider the compact set F and its boundary ∂F. The monodromy can be chosen to be the identity on the boundary. So there exists a map from the relative pair $(F, \partial F)$ to F. Let c be a relative cycle, then, since h is the identity on ∂F, c and $h(c)$ have the same boundary and thus $h(c) - c$ is a cycle on F.

Definition 4.2 *The* variation operator *of f is the map*

$$\mathrm{var}_* : H_*(F, \partial F;\mathbb{Z}) \to H_*(F;\mathbb{Z}).$$

Let $i : (F, \emptyset) \to (F, \partial F)$ be the standard inclusion, then we have a commutative diagram

$$\begin{array}{ccc} H_*(F;\mathbb{Z}) & \stackrel{h_*-id}{\longrightarrow} & H_*(F;\mathbb{Z}) \\ \downarrow i_* & \stackrel{\mathrm{var}_*}{\nearrow} & \downarrow i_* \\ H_*(F,\partial F;\mathbb{Z}) & \stackrel{h_*-id}{\longrightarrow} & H_*(F,\partial F;\mathbb{Z}) \end{array}$$

This means that we can describe the monodromy through the variation operator:

$$h_* = id + \mathrm{var}_* \circ i_*$$

Suppose that f has an isolated singularity, then we know from Corollary 3.11 that the Milnor Fibre is a wedge of spheres of dimension n and so we can concentrate on $H_n(F;\mathbb{Z})$. A particular case is that of a quadratic singularity:

Example 4.3 *Consider a Morse singularity, $f(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$. We have seen that $\mu(f) = 1$ and so $H_n(F;\mathbb{Z}) \cong \mathbb{Z}$. In fact, one can calculate that the Milnor fibre is diffeomorphic to the tangent bundle of S^n. The construction is given explicitly in [2] and [41].*

In this example we can consider the set $B_\epsilon(0) \cap f^{-1}(t)$ as $t \to 0$. We see that as $B_\epsilon(0) \cap f^{-1}(0)$ is a cone over its boundary, and hence is contractible, that the homology 'vanishes' as $t \to 0$.

Definition 4.4 *The non-trivial homology class in $H_n(F; \mathbb{Z})$ of Example 4.3 is called* the vanishing cycle *of f.*

Continuing this example, if we let ∇ denote the non-trivial class of $H_n(F, \partial F; \mathbb{Z})$ and Δ be vanishing cycle, then Picard and Lefschetz proved the following.

Theorem 4.5 ([40], [53]) *For a Morse singularity we have*

$$\operatorname{var}(\nabla) = (-1)^{n(n+1)/2}\Delta.$$

Since this theorem was first proved, the theory of vanishing cycles has been greatly developed and reframed in terms of a sheaf of vanishing cycles, see [9] for the original sheaf version and [10] and [57] for more modern exposition.

We can now generalize to more general functions. Suppose that f has an isolated singularity at $0 \in \mathbb{C}^{n+1}$. Then for a generic linear function $g : \mathbb{C}^{n+1} \to \mathbb{C}$ the function $f_\lambda : \mathbb{C}^{n+1} \to \mathbb{C}$ defined by $f_\lambda(x) = f(x) + \lambda g(x)$ will have only Morse singularities (with distinct critical values) for $0 \neq \lambda \in \mathbb{R}$.

Fix such a λ. Within the Milnor ball $B_\epsilon(0)$ we have a finite number of critical points of f_λ and this number is equal to $\mu(f)$, denote the points by $p_1, \dots, p_\mu$. Let $c_i = f_\lambda(p_i)$. In $\mathbb{C}$ we have a disc $D_\delta = \{z \in \mathbb{C} \,:\, |z| \leq \delta\}$. Over $D' = D_\delta \backslash \{c_1, \dots, c_\mu\}$ we in fact have a fibration which has its (open) fibre diffeomorphic to the Milnor fibre of f. (We use the Ehresmann fibration theorem again!) Basically, each critical point is Morse and so it will contribute one copy of $\mathbb{Z}$ to the homology of the Milnor fibre. Furthermore, the monodromy of each will determine the monodromy of f.

Let c_* be a point on the boundary of D_δ. Then, for each loop γ in D' based at c_* we have a monodromy $h_\gamma : H_n(F; \mathbb{Z}) \to H_n(F; \mathbb{Z})$. We get a homomorphism from $\pi_1(D')$ to $\operatorname{Aut} H_n(F; \mathbb{Z})$, the group of automorphisms of $H_n(F; \mathbb{Z})$.

Definition 4.6 *The* monodromy group *of f is the image of the above homomorphism.*

Let Δ_i be the vanishing cycle associated to the critical point c_i.

Proposition 4.7 *The cycles $\Delta_1, \dots, \Delta_\mu$ form a basis of $H_n(F; \mathbb{Z})$.*

We can now define an intersection pairing on $H_n(F; \mathbb{Z})$. Suppose that N is an oriented compact $2n$-manifold with boundary ∂N such that the integer homology and cohomology of N has no torsion.

There exists the Poincaré duality map $\mu : H_n(N, \partial N) \to H^n(N)$ and we also have the standard inclusion map $i : (N, \emptyset) \to (N, \partial N)$ which induces a homomorphism from $H_n(N)$ to $H_n(N, \partial N)$.

By identifying $H^n(N)$ and the dual of $H_n(N)$ we have a pairing

$$\operatorname{ev}(\ ,\ ) : H^n(N) \times H_n(N) \to \mathbb{Z}$$

given by the usual evaluation for a space and its dual.

Thus, we can define the following.

Definition 4.8 *The* intersection pairing/form *is the map $\langle\ ,\ \rangle : H_n(N) \times H_n(N) \to \mathbb{Z}$ given by*

$$\langle x, y \rangle = \operatorname{ev}(\mu(i_*(x)), y).$$

Proposition 4.9 ([2]) *We have*

$$\langle \Delta_i, \Delta_i \rangle = \begin{cases} 0, & n \text{ odd}, \\ (-1)^{n(n-1)/2} 2, & n \text{ even}. \end{cases}$$

Definition 4.10 *The matrix $B = (\langle \Delta_i, \Delta_j \rangle)_{1 \leq i,j \leq \mu}$ is called the* intersection matrix of f.

From this matrix we can determine much about the monodromy of the singularity. We can read off results about the classical monodromy operator h_* and the variation operator var since we have $h_* = h_1 h_2 \cdots h_\mu$ where

$$h_i(x) = x - (-1)^{n(n-1)/2} \langle x, \Delta_i \rangle \Delta_i$$

for Δ_i a basis element. See [2] for details.

We can also describe the monodromy of direct sums of singularities:

Theorem 4.11 *For f and g with isolated singularities we can describe the monodromy and variation operators:*

$$\begin{aligned} h_{f \oplus g*} &= h_{f*} \otimes h_{g*}, \\ \mathrm{var}_{f \oplus g*} &= \mathrm{var}_{f*} \otimes \mathrm{var}_{g*} \end{aligned}$$

The theory in the case of non-isolated singularities is more complicated as one would expect. A survey can be found in [59]. However, much of the theory carries over to functions with isolated singularities on singular spaces. Details can be found in [64] and [65].

5 Stratifications of spaces

When dealing with singular spaces (and maps) one is usually faced with a choice of two methods. The first and perhaps oldest of these is to find a manifold associated to the singular space and some map between the two. A study of the manifold and the associated map will indirectly reveal information about the singular space. This is called the resolution method. The summit of achievement here is Hironaka's famous resolution theorem which relies heavily on algebraic constructions and would take us too far from our interests.

The second method in some sense gets us closer to the singularities but still relies on using the theory of manifolds. Basically, the space is partitioned into manifold subsets, (i.e., each manifold constitutes a subset). This is called the stratification method. Different conditions on how the manifolds meet one another give rise to different types of stratification, e.g., Whitney (the most common which we shall describe below), Bekka, A_f, and logarithmic. See [63] for how the various stratifications are related to each another.

Definition 5.1 *Let X be a closed subset of a smooth manifold M and let X be decomposed into disjoint pieces S_i called strata. Then the decomposition is called a* Whitney Stratification *if the following conditions are met.*

(i). Each stratum is a locally closed smooth submanifold of M.

(ii). $S_i \cap \mathrm{Closure}(S_j) \neq \emptyset$ if and only if $S_i \subseteq \mathrm{Closure}(S_j)$ for strata S_i, S_j $i \neq j$; this is called the frontier condition *and we write $S_i < S_j$.*

(iii). Whitney Condition (a): If $x_i \in S_a$ is a sequence of points converging to $y \in S_b$ and $T_{x_i}(S_a)$ converges to a plane τ (all this considered in the appropriate Grassmannian), then $T_y(S_b) \subseteq \tau$.

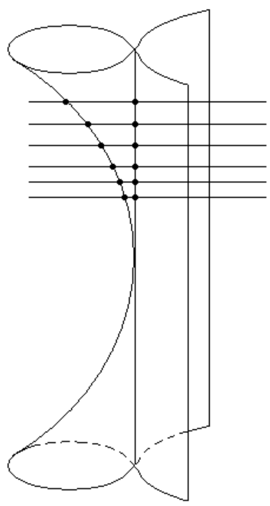

Figure 3: A space to Whitney stratify

(iv). Whitney Condition (b): If $x_i \in S_a$ *converges to* $y \in S_b$*,* $y_i \in S_b$ *also converges to* y*,* l_i *denotes the secant line between* x_i *and* y_i*, and* l_i *converges to* l*, then* $l \subseteq \tau$*.*

Note that (b)⇒(a). The condition (a) is stated explicitly since it is a condition that many stratifications satisfy and is thus very useful. If X is a closed subanalytic subset of an analytic manifold, then X can be Whitney stratified, hence complex varieties, semi-analytic spaces, etc., can be Whitney stratified.

In Figure 3 we can see that we can stratify the space by taking the z-axis as a stratum. However, this is not a Whitney stratification. The singular point that is not a transverse crossing has to be a stratum. To see this consider a family of horizontal lines as in the figure. This will converge to a line perpendicular to the z-axis so violates the (b) condition.

A Whitney stratified space can be triangulated (see [16]) and is locally topologically trivial along the strata. Also, one has a conic structure theorem like Theorem 3.1: For any point x and a small enough sphere S_ϵ centred at x, the cone of $X \cap S_\epsilon$ is homeomorphic to $X \cap B_\epsilon$.

The local topological triviality and the conic structure theorem both follow from the First Thom-Mather Isotopy Lemma. Perhaps, the most important lemmas in the applications of stratification theory are the Thom-Mather Isotopy Lemmas. The first concerns spaces and can be considered a direct generalization of the Ehresmann Fibration Theorem. Recall that the idea of the latter is that, at a non-critical value, a map on a manifold is in fact a fibration. The First Thom-Mather Isotopy Lemma essentially requires that the map is submersion on all the strata to produce a stratum preserving homeomorphism. Thus, it is very useful in proving that various spaces are homeomorphic.

The second lemma allows us to fibre certain mappings: it gives sufficient conditions for maps in a family of maps over $\mathbb{R}^p$ to be topologically right-left equivalent to one another. (Two maps $f : \mathbb{R}^n \to \mathbb{R}^p$ and $g : \mathbb{R}^n \to \mathbb{R}^p$ are *topologically right-left equivalent* if there exist homeomorphisms $h : \mathbb{R}^n \to \mathbb{R}^n$ and $k : \mathbb{R}^p \to \mathbb{R}^p$ such that $f \circ h = k \circ g$.)

Proof of both lemmas for Whitney spaces can be found in [15] and in the unpublished manuscript [46]. For Bekka (also known as (c)-regular) stratifications they are proved in [5]. We now outline the basic idea of proof. In differential topology we can find homeomorphisms by integrating vector fields on manifolds. For stratified spaces we find conditions so that we can define vector fields on the strata that when integrated give the required homeomorphisms. The surprising fact is that the vector fields do not have to be continuous when considered as a whole on the Whitney space. The method is very technical and the proof of the theorem occupies about a quarter of [15]. The original - unpublished - version is still readable, see [46]. More modern versions with some extra generalizations can be found in [54] and [63].

First Thom-Mather Isotopy Lemma

Suppose M and P are analytic manifolds. Let $X \subseteq M$ be a Whitney stratified subset.

Definition 5.2 *A map $f : X \to P$ is a stratified submersion if $f|A$ is a submersion for all strata A in X.*

Theorem 5.3 (Thom-Mather First Isotopy Lemma) *Let $f : X \to P$ be a proper stratified submersion. Then f is a locally trivial fibration such that the homeomorphisms are stratum preserving.*

As an application we can prove a Milnor fibration type theorem, first proved by Lê in [39]. We need a definition from there which clarifies what we mean by a function on a singular space having a singularity.

Definition 5.4 *Suppose that $f : X \to \mathbb{C}$ is a complex analytic function with a Whitney stratification $\mathcal{S}$ and that X can be locally embedded in $\mathbb{C}^n$ at $x \in X$ for some n.*

We say that f has an isolated stratified singularity at *x if there exists $\epsilon > 0$ such that $f|B^\circ_\epsilon(x) \cap (A\backslash\{x\}) \to \mathbb{C}$ is a submersion for all $A \in \mathcal{S}$ where $B^\circ_\epsilon(x)$ is an open ball in $\mathbb{C}^n$ of radius ϵ centred at x.*

We can now produce the general Milnor fibration set-up for singular spaces.

Proposition 5.5 ([39]) *Suppose X is a complex analytic space and $f : X \to \mathbb{C}$ has an isolated stratified singularity at $x \in f^{-1}(0)$. Then, there exists an embedding of X into $\mathbb{C}^n$ and real numbers ϵ and δ such that the map*

$$\phi : B_\epsilon(x) \cap X \cap f^{-1}(D^*_\delta) \to D^*_\delta$$

induced by f is a locally trivial fibration over the punctured disc $D^*_\delta = \{y \in \mathbb{C} :\ 0 < |y| < \delta\}\backslash\{0\}$. Here $B_\epsilon(x)$ is a small open or closed ball about x of radius ϵ in $\mathbb{C}^n$. In the closed case we produce a fibration on the boundary of $B_\epsilon(x) \cap X \cap f^{-1}(D^*_\delta)$ as well.

The fibration is sometimes referred to as the Milnor fibration. Just as in the case of a function on a complex manifold, often the distinction between the open and closed version of the fibres is blurred since the two spaces are homotopically equivalent.

The complex link

In the case where f is a general (complex) linear function, one gets the *complex link of x*. This is a very powerful space which contains a lot about the behaviour of the set X near to x.

Definition 5.6 *Let x be a point of $X \subseteq \mathbb{C}^n$ a complex analytic set. Then, for a suitably generic linear function $L : \mathbb{C}^n \to \mathbb{C}$ the* complex link *of x is*

$$\mathcal{L}_x = B_\epsilon(x) \cap X \cap L^{-1}(t)$$

where $t \neq 0$ and ϵ are sufficiently small.

Now take a manifold N transverse to the stratum A containing x such that $N \cap X = \{x\}$.

Definition 5.7 *The complex link of x in the set $X \cap N$ is called the* complex link of the stratum A *and is denoted $\mathcal{L}_A$.*

As one might imagine the topological type of $\mathcal{L}_A$ does not depend on the choice of $x \in A$ and the choices of ϵ and t when these are small.

For complex spaces, the complex link of strata turns out to be the fundamental building blocks of the space and analysis of them is vital to the use of Stratified Morse Theory in Section 6.

Few results are known that describe the complex link of strata in general settings. The following is very useful and quite general. This was proved by Lê following Hamm's work for the similar situation of real links.

Theorem 5.8 *Suppose that X is defined at x in $\mathbb{C}^n$ by r equations. Then, the complex link of the stratum containing x is $(n - r - 2)$-connected.*

Corollary 5.9 *Suppose that X is a local complete intersection at x, i.e., the number of defining equations equals the codimension in $\mathbb{C}^n$ of X at x, then the complex link of the stratum A is homotopically equivalent to a wedge of spheres of dimension* $\dim X - \dim A - 1$.

There are further examples.

Examples 5.10 *(i). Suppose that $f : N \to P$ is a complex analytic map between complex manifolds such that $\dim N < \dim P$ and that f is stable everywhere (see [66]) and the corank of the differential is less than or equal to 1 at all points in N. (In this case we say that f has corank 1 even though we include the case that the corank may be zero.)*

We can stratify the image by stable type and this is the canonical Whitney stratification, see [14]. Then the complex link of a stratum A in the image is homotopically equivalent to a single sphere of dimension $k \dim N - (k - 1) \dim P - \dim A - 1$ where k is determined precisely by the stable type of the germ at any point of A. See [29].

(ii). Let $F : \mathbb{C}^2 \to \mathbb{C}^4$ be defined by $F(x, t) = (x^2, x^3 + tx, tx^3, t)$. The image of this map has an isolated singularity at the origin of $\mathbb{C}^4$ and is defined in $\mathbb{C}^4$ by no fewer than four equations, so it is not a complete intersection. The link $\mathcal{L}$ of the origin is homeomorphic to the image of the map $f_t(x) = F(x, t)$ for any $t \neq 0$. Since, for $t \neq 0$, f_t is a proper injective immersion the image is homeomorphic to $\mathbb{C}^2$, which is contractible.

(iii). Suppose that X_n is the complex analytic set in the set of $2 \times n$ matrices given by the matrices of rank 1. Then X_n is $n + 1$ dimensional, embedded in $\mathbb{C}^{2n}$ and has an isolated singularity: the matrix of rank zero. The set X_n can be seen to be an analytic set in $\mathbb{C}^{2n}$ via taking all the 2×2 minors of the following matrix, where coordinates on $\mathbb{C}^{2n}$ are given by z_i,

$$\begin{bmatrix} z_1 & z_2 & \dots & z_n \\ z_{n+1} & z_{n+2} & \dots & z_{2n} \end{bmatrix}.$$

Using this description it is possible to calculate that the complex link of the origin is homeomorphic to $\mathbb{CP}^n$.

We have seen that wedge-of-spheres theorems occur regularly for Milnor fibres and we will see in the next section that complex links are the building blocks of complex analytic spaces. We now show a general theorem that describes the Milnor fibre on a singular space in terms of a wedge of suspensions of complex links, see [61]. Hence in the case where the complex links are wedges of spheres, for example, X is a complete intersection, we can deduce the Milnor fibre is a wedge of spheres.

First recall that the *suspension* of a topological space Z, denoted ΣZ is the join of Z and two disjoint points. The repetition of this process is denoted $\Sigma^k Z$, where $\Sigma^1(Z) = \Sigma Z$.

Theorem 5.11 (General wedge theorem, [61]) *Let X be a complex analytic space with a Whitney stratification such that $f : X \to \mathbb{C}$ has an isolated stratified singularity.*

Then F_f is homotopically equivalent to

$$\bigvee_A \bigvee_{\mu_A(f)} \Sigma^{\dim_{\mathbb{C}} A} \mathcal{L}_A$$

where A runs all over all strata A such that $x \in \overline{A}$, the closure of A, and $\mu_A(f)$ is the number of copies to be taken (which depends on f).

This then encompasses many of the theorems we have met. For example, if $X = \mathbb{C}^{n+1}$ and f has an isolated singularity, then the complex links are empty and $\Sigma^{n+1}\emptyset = S^n$ (because $\Sigma\emptyset = S^0$). Thus, we recover the original Milnor fibre theorem, Corollary 3.11.

Second Thom-Mather Isotopy Lemma

The second isotopy lemma is a relative version of the first, rather than using it to fibre a singular space, we fibre a map between two singular spaces.

Suppose M, N and P are analytic manifolds. Let $X \subseteq M$ and $Y \subseteq N$ be Whitney stratified subsets.

Definition 5.12 *Suppose X and Y are Whitney stratified spaces in the analytic manifolds M and N and $f : X \to Y$ is the restriction of a smooth map $F : M \to N$. Then, the map is called* stratified *if f is proper and if for any stratum $A \subseteq Y$ the preimage $f^{-1}(A)$ is a union of strata and f takes these strata submersively to A.*

If X and Y are complex analytic and f is complex analytic, then it is possible to stratify X and Y into complex analytic strata so that f is stratified. See [17] I.1.7.

Just as it was necessary to impose further conditions on the strata of the space to get the First Isotopy Lemma we need to impose another important condition on the map for the Second. This is the Thom A_f condition:

Definition 5.13 *Let $f : X \to Y$ be a stratified map. Then f is a* Thom A_f map *if for every pair of strata $B < A$ we have the following.*

(i). $f|A$ and $f|B$ have constant rank.

(ii). A is Thom regular over B: If $a_i \in A$ is a sequence of points converging to $b \in B$ such that $\ker d_{a_i}(f|A)$ converges to a plane T then $\ker d_b(f|B) \subseteq T$.

This condition is important in its own right. The definition of (c)-regularity (or Bekka stratifications) involves the Thom condition, see [5]. It is generally thought that this type of stratification, which includes Whitney stratifications, is the 'correct' type of stratification for the study of maps between spaces.

Theorem 5.14 (Thom-Mather Second Isotopy Lemma) *Let $F : X \to Y$ be a proper Thom A_F map and let $f : Y \to P$ be a proper stratified submersion. Then $F : X \to Y$ is locally topologically trivial over P.*

That is, for every point $p \in P$ there exists a neighbourhood V of p, such that for every $q \in V$, $F : (f \circ F)^{-1}(q) \to f^{-1}(q)$ and $F : (f \circ F)^{-1}(p) \to f^{-1}(p)$ are topologically right-left equivalent by stratum preserving homeomorphisms.

The Thom-Mather Isotopy Lemmas are important ingredients in the study of the topological stability of maps, see [15], and for more recent progress and a highly detailed exposition see [12].

6 Stratified Morse Theory

Classical Morse Theory

Morse theory has a long history, going back before it even acquired that name. It is fairly obvious that given a topological space X and a continuous map $f : X \to \mathbb{R}$ then we can study the topology of X by seeing how it changes as we take the preimages under f of the set $(-\infty, a]$. Let $X_a = f^{-1}([-\infty, a])$; we can define a critical value v of f to be one such that $X_{v-\varepsilon}$ is not homeomorphic to $X_{v+\varepsilon}$, for any small ε. The idea is to find a suitable f so that, for instance, we have a finite number of critical values and can find out what happens as we pass them.

Of course such a situation is too general and we have to restrict to situations where X has some extra structure, for instance a nonsingular projective complex algebraic curve. According to Fulton in [13] Riemann had the following theorem: Suppose $f : X \to \mathbb{CP}^1$ is a meromorphic function on a smooth projective curve with n sheets and w simple branch points, then the genus g of X is given by $w = 2g + 2n - 2$.

Various generalizations were given until Morse eventually arrived at the following theorem:

Theorem 6.1 *Suppose $f : X \to \mathbb{R}$ is a 'sufficiently general' C^∞ function on the compact real manifold X. Then the topology of X_a changes only when we pass a critical value, which in this case is the image of a point where the differential of f is zero. Furthermore $X_{v+\varepsilon}$ is homotopically equivalent to $X_{v-\varepsilon}$ with a cell attached and the dimension of this cell is equal to the dimension of the space upon which the Hessian of f is negative definite (this is called the* index *of f).*

This is a truly great result, its effectiveness in the 20th Century can be seen in Smale's work on h-cobordism and the higher dimensional Poincaré conjecture, René Thom's work, Bott periodicity, etc. Classical Morse theory was explained very well in Milnor's classic book [48] and so I shall not explain it further here.

A stratified version

Now, we can easily ask if such a theory is possible for singular spaces rather than manifolds. Goresky and Macpherson answered this positively with their *Stratified Morse Theory.*

The idea here is that we stratify our singular space into manifolds and put a function on the space so that function is a Morse function on the manifolds and

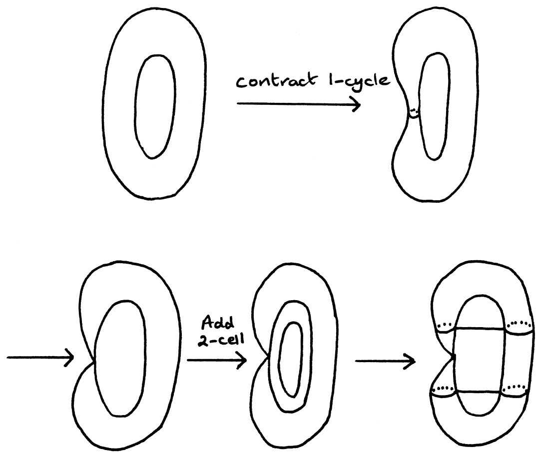

Figure 4: A stratified space from Goresky and MacPherson [17].

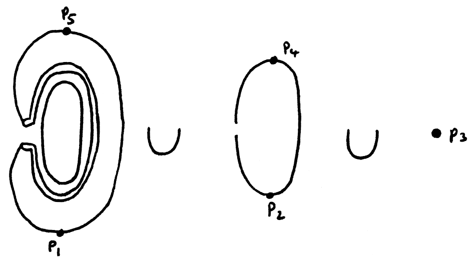

Figure 5: Stratification of torus with critical points.

so that the function behaves in a non-degenerate way with respect to how the manifolds meet each other. Then just as in Morse Theory we can describe how the topology of the singular space changes as we pass through critical points. The most developed method of stratification is Whitney stratification and, in fact, Goresky and Macpherson's original work was only for these stratifications but work of King, [35], and of Hamm, [23], allows us to use other stratifications, such as (c)-regular.

Let's see with an example how Stratified Morse Theory allows us to build up the topology of a space from simple building blocks. This example is taken from the introduction of [17]. The example used in illustrating Classical Morse theory is the torus with a height function. It is well known that part of the homology of the torus is generated by two circles. So to change the topology and make the space singular we can collapse one circle to a point and glue a 2-cell to the other as shown in Figure 4. The stratification is shown in Figure 5. The Morse function will be the height function h which gives the critical points $p_1, \ldots, p_5$ labelled. Note that the stratum of dimension zero is needed otherwise the stratification would not satisfy the Whitney (b) condition.

As the height increases we can see that the topology of the space changes as we pass the critical values of h restricted to the various strata. Furthermore the change in topology depends only upon a small enough neighbourhood of the critical point. The space we attach to $X_{v-\varepsilon}$ to get $X_{v+\varepsilon}$ is called the *Morse data*.

The fact that is probably not obvious is that the Morse data is a product of two spaces. If we restrict our attention to the stratum containing the critical point

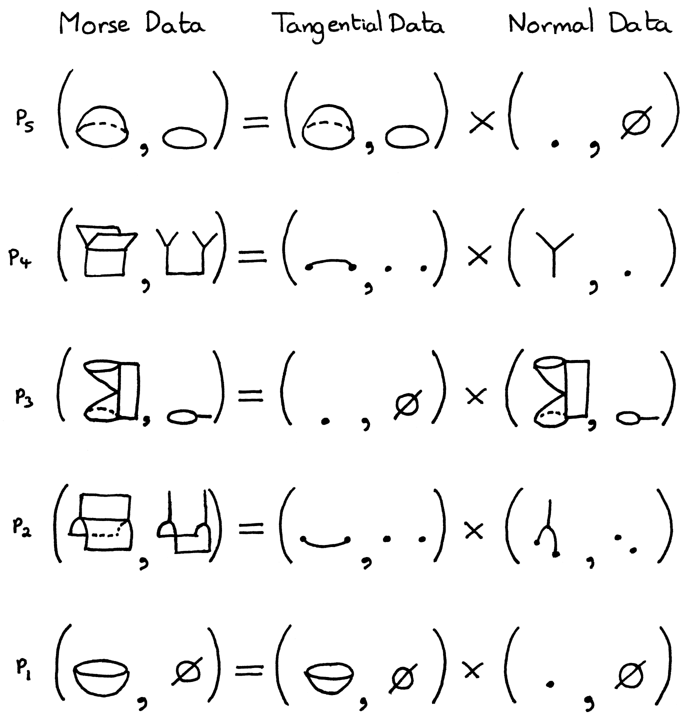

Figure 6: Product structure of the local Morse data.

then by classical Morse theory we add a cell to the manifold to get the new space. However, due to the Whitney conditions, along the stratum the neighbourhood is a product. Thus one can see that the Morse data should be a product of the classical Morse data and a space associated to a slice transverse to the stratum. Figure 6 shows the product structure of the Morse data for our example.

Morse functions

As in any Morse type theory we have to decide which functions are suitable and how common they are. Clearly the functions should satisfy the usual properties of Morse functions from classical theory: nondegenerate critical points with non-coincident critical values when restricted to the manifolds. The extra condition, (like most in stratification theory) relates to what is occurring in the normal direction to the manifold containing the critical point. First we need a definition:

Definition 6.2 *Suppose X is a Whitney stratified space in the smooth manifold M and p is point in the stratum A of X. Then $Q \subseteq T_pM$ is a* generalized tangent space *at p to X if Q is the limit of tangent planes for a sequence of points converging to p. Hence, for example, T_pA is a generalized tangent space.*

Definition 6.3 *A* Morse function *f on the Whitney stratified set $X \subset M$ is a function that is the restriction of a smooth function F on M such that*

(i). $f = F|X$ is proper;

(ii). for each stratum A of X the critical points of $f|A$ are nondegenerate and the critical values are distinct;

(iii). for every critical point $p \in A$ and for every generalized tangent plane Q at p, $dF(Q) \neq 0$ except if $Q = T_p(A)$.

Under mild restrictions there is a plentiful supply of Morse functions on a particular space.

Theorem 6.4 *Suppose X is a Whitney stratified subanalytic subset of the analytic manifold M. Then the functions $F : M \to \mathbb{R}$ that restrict to stratified Morse functions on X form an open and dense subset of the space of smooth functions. If $M = \mathbb{R}^n$, then the function given by restriction to X of the distance from a generic point in M is an example of a stratified Morse function.*

Morse data

Suppose X is a Whitney stratified space in the analytic manifold M and $f : X \to \mathbb{R}$ is a stratified Morse function with a critical $p \in A$, with critical value v, where A is a stratum of dimension n. Let N be a submanifold of M transverse to A with $N \cap A = \{p\}$ and let B be a small enough ball in M centred at p.

Define *Tangential Morse Data*, TMD, to be the space

$$\mathrm{TMD} = (D^\lambda \times D^{n-\lambda}, S^{\lambda-1} \times D^{n-\lambda})$$

where λ is the classical Morse index of $f|A$ at p, D^λ is a disc of dimension λ and $S^{\lambda-1}$ its boundary.

Define *Normal Morse Data*, NMD, to be the space

$$\mathrm{NMD} = (X \cap B \cap N) \cap (f^{-1}([v-\varepsilon, v+\varepsilon]), f^{-1}(v-\varepsilon)).$$

The space $X \cap B \cap N \cap f^{-1}(v-\varepsilon)$ is called *the halflink of f at p* and is denoted l^-.

Define the *Morse data* to be the space $\mathrm{MD} = \mathrm{TMD} \times \mathrm{NMD}$ where

$$(A, B) \times (C, D) = (A \times D, A \times D \cup B \times C).$$

Recall that $X_a = \{x \in X \,:\, x \in f^{-1}((-\infty, a]\}$.

Theorem 6.5 (Fundamental Theorem of Stratified Morse Theory) *With the above conditions*

$$X_{v+\varepsilon} \simeq X_{v-\varepsilon} \cup \mathrm{MD}$$

where the union of a space with a pair is taken to mean the attachment of the first space via the second.

Thus the Morse data for f is a product of the Tangential and Normal Morse Data.

The proof of the main theorem of Stratified Morse Theory, (the Morse Data of a function is a product of the Morse data for the stratum and the Morse data for the normal slice), occupies a large part of Goresky and MacPherson's book [17]. In effect they used repeated application of the First Thom-Mather Isotopy Lemma which has been disguised in a more useful (for their purposes) technique called Moving the Wall.

In subsequent years the proof has been simplified and generalized to stratified spaces other than Whitney, first by King in [35] and then by Hamm in [23] (which dealt with some of the weaknesses in [35]).

The method employed arises from the 'direct sum' idea in creating new singularities (see *New Milnor fibres from old* in Section 3). One can define the Morse data of an arbitrary continuous function as the pair $X \cap B \cap f^{-1}([v-\varepsilon, v+\varepsilon], v-\varepsilon)$ where B is a small ball around a critical point, v is the critical value and ε is a small number.

King's idea is that if we have a two functions $f_i : X_i \to \mathbb{R}$, $i = 1, 2$, where the functions and spaces are restricted to 'good' categories (Whitney stratified spaces and stratified Morse functions form a good category), then the Morse data for the function $f_1 \oplus f_2 : X_1 \times X_2 \to \mathbb{R}$ given by $(f_1 \oplus f_2)(x_1, x_2) = f_1(x_1) + f_2(x_2)$ is homeomorphic to the product of the Morse data for f_1 and f_2. The idea then is to show that for a stratified Morse function on the Whitney stratified space X we can 'split the function over' the product of the stratum and a normal slice.

This process can be exemplified using classical Morse theory. The main theorem arises from the Morse lemma: at a critical point of a Morse function f there is a choice of local coordinates such that

$$f(x_1, \dots, x_n) = \sum_{i}^{k} x_i^2 - \sum_{k+1}^{n} x_i^2.$$

The Morse data for $g(x_1, \dots, x_k) = \sum_i^k x_i^2$ and $h(x_{k+1}, \dots, x_n) = -\sum_{k+1}^n x_i^2$ are $(D^k, \emptyset)$ and (D^{n-k}, S^{n-k-1}) respectively. Since $f = g + h$ the Morse data for f is homeomorphic to

$$(D^k \times D^{n-k}, D^{n-k} \times \emptyset \cup D^k \times S^{n-k-1}).$$

This is equal to

$$(D^k \times D^{n-k}, D^k \times S^{n-k-1}).$$

Thus we recover the classical Morse Theory result.

The complex analytic setting

In obtaining the Morse data in the stratified case there are two objects to describe: the tangential data and the normal data. Finding the tangential data is merely the determination of a number: the number of negative eigenvalues of the Hessian. The normal data is more difficult to work out; it depends on the singularities of X and the function f.

Even if X is a (stratified) manifold this can prove problematic. However, for complex manifolds the Morse indices of the standard distance function are bounded above by the complex dimension rather than just the real dimension. In a similar way the normal data becomes simpler for complex analytic spaces. In fact, homotopically speaking, the normal Morse data does not depend on the function and hence to determine it we can investigate the normal data for a linear function on X. Better than that, the data is homeomorphic to a product of an interval and the intersection of X with a generic linear complex form. In other words, the complex link of the stratum! This allows us to prove theorems using induction as the complex link is a complex space of dimension one lower than X.

The main theorem for Stratified Morse Theory on a complex analytic space is the following.

Theorem 6.6 *Suppose that f is* any *stratified Morse function on X. Then the normal Morse data does not depend on the function. Furthermore,* NMD $\simeq (\mathrm{Cone}(\mathcal{L}), \mathcal{L})$.

It should be noted that up to homotopy the above spaces do not depend on any choices involved, eg the choice of ε, metric on M, normal slice, etc.

7 Rectified Homotopical Depth.

We now give a simple introduction to the notion of rectified homotopical depth (abbr. rhd). Rectified homotopical depth was introduced by Grothendieck in [20] to measure the failure of the Lefschetz hyperplane section theorem for singular spaces. The original theorem is the following.

Theorem 7.1 (Lefschetz Hyperplane theorem) *Suppose that* $X \subseteq \mathbb{CP}^m$ *is a non-singular projective algebraic variety and* H *a hyperplane. Then* $\pi_i(X, X \cap H) = 0$ *for* $i < \dim(X)$.

Lefschetz said this 'planted the harpoon of algebraic topology into the body of the whale of algebraic geometry', see [40] page 13.

Our interest in rhd arises from the fact that measuring the rhd for a complex analytic space tells us something about its Normal Morse data, (which as we have said, is independent of the Morse function). We shall see that if we replace the manifold X by a singular space, then we can replace $\dim(X)$ in the theorem with $\mathrm{rhd}(X)$.

Let X be a complex analytic space with stratification $\mathcal{S}$. For any stratum A in $\mathcal{S}$ let L_A denote the real link of A and let $\mathcal{L}_A$ denote the complex link. In our set up rhd keeps track of the vanishing of homotopy groups for these spaces.

We actually define the *rectified homotopical depth* of X, denoted $\mathrm{rhd}(X)$ using the following proposition.

Proposition 7.2 *The following are equivalent:*

(i). $\mathrm{rhd}(X) \geq n$,

(ii). $\pi_i(\mathrm{Cone}(L_A), L_A) = 0$ *for* $i < n - \dim_{\mathbb{C}} A$ *for all strata* $A \in \mathcal{S}$,

(iii). $\pi_i(\mathrm{Cone}(\mathcal{L}_A), \mathcal{L}_A) = 0$ *for* $i < n - \dim_{\mathbb{C}} A$ *for all strata* $A \in S$.

Thus the rhd of X is the largest number for which the second two statements hold. It is independent of the stratification chosen for the space, see [24].

The number $\mathrm{rhd}(X)$ is bounded above by the dimension of X because both types of link of the largest stratum are equal to the empty set and $\mathrm{Cone}(\emptyset)$ is defined to be a point. Thus $\pi_0(\mathrm{Cone}(\emptyset), \emptyset) = \pi_0(\mathrm{point}, \emptyset) \neq 0$. In particular, if X is non-singular, then $\mathrm{rhd}(X) = \dim_{\mathbb{C}}(X)$.

Similarly we can define *rectified homological depth*, $\mathrm{rHd}(X; \mathbb{Z})$ by replacing the relative homotopy groups in the definition by relative homology groups. It is also possible to make a definition of rectified homological depth, $\mathrm{rHd}(X; G)$, for any coefficient group G.

We have the following lemma:

Lemma 7.3 *For a field* F,

$$\mathrm{rhd}(X) \leq \mathrm{rHd}(X; \mathbb{Z}) \leq \mathrm{rHd}(X; F) \leq \dim_{\mathbb{C}} X.$$

The first relation is proved by using Hurewicz's theorem. The second by the universal coefficient theorem. As above the last relation is simple.

Theorem 7.4 *Let* $f : X \to \mathbb{R}$ *be a stratified Morse function with a critical point at* p *in the stratum* A *with critical value* v. *Let* $X_b = f^{-1}([-\infty, b])$. *Let* λ *be the Morse index at* p *and* ε *a sufficiently small number, then*

(i). $\pi_i(X_{v+\varepsilon}, X_{v-\varepsilon}) = 0$ *for* $i < (\lambda - \dim_{\mathbb{C}} A) + \mathrm{rhd}(X)$,

(ii). $H_i(X_{v+\varepsilon}, X_{v-\varepsilon}) = 0$ *for* $i < (\lambda - \dim_{\mathbb{C}} A) + \mathrm{rHd}(X)$.

The second is easier to prove. Using excision we get

$$H_i(X_{v+\varepsilon}, X_{v+\varepsilon}) \cong H_i(\mathrm{NMD}, \partial\,\mathrm{NMD}).$$

But $(\mathrm{NMD}, \partial\,\mathrm{NMD})$ is equal to the product $(D^\lambda, \partial D^\lambda) \times (\mathrm{Cone}(\mathcal{L}_A), \mathcal{L}_A)$. The product theorem for homology gives the vanishing of homology that is required.

For (i) more advanced and less well known techniques need to be used but the spirit is the same. One can construct a proof using [17] II.4.

An important theorem, which is essentially a local Lefschetz type theorem, was proved by Hamm and Lê. See [24] 3.2.1.

Theorem 7.5 *Suppose that X is a complex analytic space and Y is a subspace defined set theoretically by no more than r equations, then*

$$\mathrm{rhd}(Y) \geq \mathrm{rhd}(X) - r.$$

The proof actually follows from reasoning similar to the proof of Theorem 5.8.

As a corollary of this we are able to find the rhd of a very large class of complex analytic spaces.

Corollary 7.6 *Suppose that X is a local complete intersection. Then* $\mathrm{rhd}(X) = \dim_{\mathbb{C}} X$.

Grothendieck's original intention for rectified homotopical depth was that is was analogous to the notion of depth from commutative algebra. For example, one can see that for regular rings and complete intersection rings that depth equals the dimension of the ring. For rectified homotopical depth we have that manifolds (equivalent to regular rings in the analogy) and local complete intersections have $\mathrm{rhd}(X) = \dim X$.

Rings with maximal depth are called Cohen–Macaulay and have many interesting properties. Similarly, spaces with maximal rectified homotopical depth has good properties - since, almost by definition, their complex links are wedges of spheres in middle dimension.

An interesting unexplored topic is the analogy with Gorenstein rings. To some extent this was tackled in Goresky and MacPherson's original work on Poincaré duality for singular spaces but it would be good to pursue the analogy further from the perspective of rectified homotopical depth.

Example 7.7 *An example of a space that does not have* $\mathrm{rhd}(X) = \dim_{\mathbb{C}} X$*: Let $X = \mathbb{C}^2 \cup \mathbb{C}^2$ be two copies of $\mathbb{C}^2$ in $\mathbb{C}^4$ that intersect transversally at the origin. Stratify X by taking the origin as one stratum and the complement of the origin in X as the other. It is obvious that the complex link of the origin, $\mathcal{L}$, is a disjoint union of two discs. Thus $\pi_0(\mathrm{Cone}(\mathcal{L}), \mathcal{L}) = 0$ but $\pi_1(\mathrm{Cone}(\mathcal{L}), \mathcal{L}) \neq 0$. Therefore* $\mathrm{rhd}(X) = 1 < \dim_{\mathbb{C}} X = 2$.

Another theorem of interest concerns the notion of perverse sheaves, see [4] for the definition. Let $\mathbb{C}^\bullet_X[\dim_{\mathbb{C}} X]$ denote the sheaf complex that has the constant sheaf of complex numbers in the $\dim_{\mathbb{C}} X$ position and zero in other degrees.

Theorem 7.8 ([24]) *We have:* $\mathrm{rHd}(X; \mathbb{Q}) = \dim_{\mathbb{C}} X$ *if and only if* $\mathbb{C}^\bullet_X[\dim_{\mathbb{C}} X]$ *is perverse in the sense of Bernstein-Beilensen-Deligne.*

The final theorem of this section shows how rhd can be used to give a Lefschetz theorem. The proof is a simple example of the type of technique used in proving Lefschetz type theorems.

Theorem 7.9 *Let X be a complex projective variety in $\mathbb{CP}^N$. Suppose H is a hyperplane in $\mathbb{CP}^N$ that does not contain all of X, then*

$$\pi_i(X, X \cap H) = 0 \text{ for } i < \mathrm{rhd}(X - H).$$

We give a proof here since almost all Lefschetz type theorems can be proved using the outline of this proof.

Proof. Identify $\mathbb{CP}^N - H$ with $\mathbb{C}^N$. For a generic point p in $\mathbb{C}^N$ the function $\phi(z) = ||z-p||^2$ is a stratified Morse function on $X \cap \mathbb{C}^N$. This function has a finite number of critical points and the Morse index for a critical point on the stratum A is not greater than $\dim_{\mathbb{C}} A$. For some large enough R all critical point of ϕ are contained in the set $\phi^{-1}([0,R])$. We consider the Morse function $-\phi$ on the set $X \cap \mathbb{C}^N$. The Morse indices are bounded below by $2\dim_{\mathbb{C}} A - \dim_{\mathbb{C}} A = \dim_{\mathbb{C}} A$. Let $X_a = (-\phi)^{-1}(-\infty, a)$. By passing through all critical points of $-\phi$ we build up X from X_{-R} using stratified Morse theory. The critical points of ϕ all lie outside $X \cap H$ and thus the connectivity of the complex link of the stratum depends only on $\mathrm{rhd}(X-H)$ and not $\mathrm{rhd}(X)$.

By Theorem 7.4 we get $\pi_i(X_{v+\varepsilon}, X_{v-\varepsilon}) = 0$ for $i < (\lambda - \dim_{\mathbb{C}} A + \mathrm{rhd}(X-H))$ for any critical value v, where λ is the Morse index of the critical point. As $\lambda \geq \dim_{\mathbb{C}} A$, the pair is therefore $(\mathrm{rhd}(X-H)-1)$-connected. So (X, X_{-R}) is $(\mathrm{rhd}(X-H)-1)$-connected.

X can be triangulated with $X \cap H$ being a subtriangulation. Thus there exists a neighbourhood U of $X \cap H$ that retracts onto $X \cap H$. For some $R' > R$ the space $X_{-R'}$ is a subset of U and since the interval $[-R', -R]$ contains no critical points for $-\phi$, by Thom's First Isotopy Lemma, X_{-R} retracts onto $X_{-R'}$. Thus in the chain of inclusions,

$$X \cap H \subseteq X_{-R'} \subseteq U \subseteq X_{-R}$$

the composition of any two inclusions induces an isomorphism on homotopy groups. This implies that the natural map $\pi_*(X \cap H) \to \pi_*(X_{-R})$ is an isomorphism. By examining the long exact sequence arising from the triple $(X, X_{-R}, X \cap H)$ we arrive at $\pi_i(X, X \cap H) = 0$ for $i < \mathrm{rhd}(X-H)$ as stated. □

This theorem also exemplifies the flavour of proofs using Stratified Morse Theory. Essentially the process involves taking a stratified Morse function and using it to build up one space from another. Something about the function allows us to bound the indices from above or below and the rhd hypothesis allows us to say something about the connectivity of the normal Morse data. Thus we get a theorem involving rhd which in effect tells us nothing practical until we 'attack' with an example where we have calculated the rhd. For example, we have the following.

Corollary 7.10 ([48] Theorem 7.4.) *Suppose that $X \cap H$ contains the singular locus of X. Then*

$$\pi_i(X, X \cap H) = 0 \text{ for } i < \dim_{\mathbb{C}} X.$$

Since $X - H$ is nonsingular we have $\mathrm{rhd}(X-H) = \dim_{\mathbb{C}}(X-H) = \dim_{\mathbb{C}} X$ and we can apply the theorem. We also have the following.

Corollary 7.11 *Suppose that X is a local complete intersection. Then*

$$\pi_i(X, X \cap H) = 0 \text{ for } i < \dim_{\mathbb{C}} X.$$

Since $X - H$ is a local complete intersection by Theorem 7.5 we have $\mathrm{rhd}(X-H) = \dim_{\mathbb{C}}(X-H) = \dim_{\mathbb{C}} X$. This of course includes the case that X is non-singular.

Once the overall method of proof has been grasped it is possible to generalize to other cases where the space is not in $\mathbb{CP}^n$. Here one also has to estimate the Morse index and this is usually done by using the concept of q-convexity. See [17], [22] and [57].

8 Relative Stratified Morse Theory.

Suppose $X \subseteq M'$ and $Z \subseteq M$ are closed Whitney stratified subsets of smooth manifolds and that $\pi : X \to Z$ is a proper surjective stratified map, i.e., π is the restriction of $\pi' : M' \to M$, a smooth map such that π maps strata of X submersively to strata of Z. This is not an unnatural set up for if X and Z are complex analytic spaces and π is a proper complex analytic map, then there exist stratifications of X and Z such that π becomes a stratified map. (See [17] page 43.) Let $f : Z \to \mathbb{R}$ be a stratified Morse function with a critical point at p.

The idea of relative stratified Morse theory is to build up the space X using the maps f and π. The composition $f \circ \pi$ is not a Morse function and approximating $f \circ \pi$ by a Morse function may not viable since we could lose estimates on the Morse indices.

Nevertheless, this set up is amenable to study. For if the interval $[a, b]$ contains no critical value of f, then X_a is homeomorphic in a stratum preserving way to X_b. This is not too difficult to prove as it relies upon stratification techniques and the First Thom-Mather Isotopy Lemma. The important result is the following:

Theorem 8.1 *Let $l^- = \pi^{-1}(Z \cap N \cap B_\epsilon(p)) \cap f^{-1}(-\varepsilon)$. There exists a map $\phi : l^- \to \pi^{-1}(p)$, see [17] page 117. If λ is the Morse index of f at the critical point p, then X_b has the homotopy type of X_a with the attachment of the pair*

$$(D^\lambda, \partial D^\lambda) \times (\mathrm{cyl}(l^- \to \pi^{-1}(p)), \pi^{-1}(l^-))$$

where cyl *denotes the mapping cylinder of ϕ and D^λ is the standard disc of dimension λ.*

The proof is given in [17].

Thus we see that the tangential Morse data does not depend on π and our main difficulty with the set up is to calculate the 'relative' Morse data.

If Z is complex analytic, then l^- is less complicated because it is homotopically equivalent to the preimage of the complex link of the stratum containing p.

If X is also complex analytic and π is a finite complex analytic map then the structure of the mapping cylinder becomes simpler. It is the disjoint union of cones over the spaces involved.

There are many as yet unexplored situations where relative Stratified Morse Theory can be applied. In [17] various relative versions of the Lefschetz Hyperplane are proved. Another application is given in [26]. The set up there is the following for the case $G = S_k$, the group of permutations on k objects. Suppose G is a finite group acting upon the complex analytic space X. The map $\pi : X \to X/G$ is a finite surjective complex analytic map which we can stratify. This map is then a Thom A_π map. The relative Morse data will inherit a G action since we can lift a vector field giving a homeomorphism on X/G to a G-equivariant vector field on X. (We can lift the vector fields as π is a Thom A_π map and these fields are G-invariant). Thus X_b is G-homotopically equivalent to the union of X_a and the relative Morse data.

Furthermore, if M is a real manifold intersecting the strata of X/G transversally, then we can lift vector fields on $(X/G) \cap M$ to G-equivariant vector fields on $X \cap \pi^{-1}(M)$.

9 Topology of images and multiple point spaces

Through most of this article our concern has been with the topology of fibres. The Implicit Function Theorem, Sard's Lemma and the Morse Lemma effectively tell us that the topology of fibres is, in some sense, easy to deal with: Most fibres are non-singular and most singular functions have isolated quadratic singularities.

But what about the topology of images? This is considerably harder to deal with. Let us first consider one of the main difficulties. This is the realisation that singularities are common for images. Consider the Whitney cross-cap of Example 2.9(ii). This is a stable map - a small perturbation will produce an equivalent singularity under local diffeomorphisms of the source and target - and so this is a perfectly natural object. However, the image has non-isolated singularities. In the case of critical points of functions or maps we had the possibility of taking a nearby fibre which was non-singular. We do not have this possibility for the Whitney umbrella. There is no natural nearby non-singular image. Furthermore, to make matters worse, the umbrella is the image of an algebraic map, but the image itself is not algebraic in the real case - it is semi-algebraic.

To show that progress can be made in describing the topology of the image of a map consider the following. Suppose that we consider a smooth map $f : N \to P$, from a closed surface N, (i.e., non-singular, compact and without boundary), to a 3-manifold P, such that the only singularities of f are the stable. The singularities that occur are the Whitney cross cap or the transverse crossing of 2 or 3 sheets. In the latter case, these are called triple points and will be isolated just as the cross caps are isolated.

Theorem 9.1 (Izumiya-Marar [32]) *Suppose that $f : N \to P$ is a stable mapping from a closed surface to a 3-manifold. Then*

$$\chi(f(N)) = \chi(N) + \frac{C(f)}{2} + T(f),$$

where $\chi(X)$ denotes the Euler characteristic of the space X, $C(f)$ is the number of cross caps and $T(f)$ is the number of triple points.

This has been generalized in a number of directions, see [28]. The point is that we look at significant features of the map.

Example 9.2 *Steiner's Roman surface: Suppose that f is the map $f : \mathbb{RP}^2 \to \mathbb{R}^3$ given by $f([x : y : z]) = [xy : xz : yz : x^2 + y^2 + z^2]$, (since $x = y = z = 0$ is impossible we have $[a : b : c : 0] \notin f(\mathbb{RP}^2)$ so the target of the map is indeed $\mathbb{R}^3$). See [1] p40 for more information.*

The map f is smooth and has stable singularities, with 6 cross caps and 1 triple point. See Figure 7. Note that three of the cross caps are hidden from view.

Here

$$\chi(R) = \chi(\mathbb{RP}^2) + C(f)/2 + T(f) = 1 + (6/2) + 1 = 5.$$

From the figure it is possible to see that the space is homotopically equivalent to a wedge of four 2-spheres.

This examples shows how we have to be careful. We have to look at the singularities of the *maps* rather than the singularities of the *image*. One can have a non-stable map such that the singularities of the image are locally homeomorphic to the singularities of the image of a stable map. For example, Steiner's Roman surface can also be given as the image of a map from S^2 to $\mathbb{R}^3$. The triple point is formed by four corners of a cube (see Example 2.10) coming together. In this case

$$\chi(S^2) + C(f)/2 + T(f) = 2 + (6/2) + 1 = 6$$

if we count the C and T by what the singularities of the image look like.

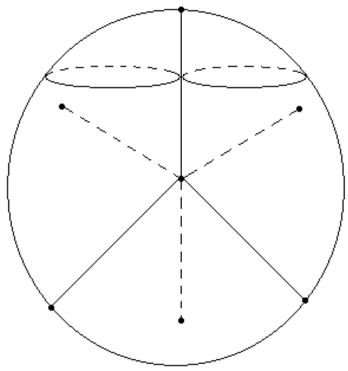

Figure 7: Steiner's Roman Surface.

Multiple Point Spaces

We shall assume now that our maps are finite and proper, i.e., each point in the target has a finite number of preimages and the preimage of a compact set is compact); for the moment we shall not assume smoothness of the map, and hence will have a continuous map $f : X \to Y$.

There are many ways of defining multiple point spaces for a finite and proper map. For example, one can define the double point set as the set of points in X where f is not injective. That is, the closure of the set $x \in X$ such that there exists $y \neq x$ such that $f(x) = f(y)$. Alternatively, some authors define the double point set as the *image* of this set.

We take a third alternative which has a number of advantages. The double point space of a map f is the closure in $X^2(= X \times X)$ of the set of pairs (x, y), with $x \neq y$, such that $f(x) = f(y)$. This first advantage of this is that often this space is, in some vague sense, less singular than that which the other definitions give. The second advantage, though this may not appear so useful at first sight, is that this space has more symmetry – the group of permutation on 2 objects acts on X^2 by permutation of copies.

We can generalize this so that the k^{th} *multiple point space of a map* is the closure of the set of k-tuples of pairwise distinct points having the same image:

Definition 9.3 *Let $f : X \to Y$ be a finite map of topological spaces. Then, the k^{th}* multiple point space of f*, denoted $D^k(f)$, is defined to be*

$$D^k(f) := \text{closure}\{(x_1, \dots, x_k) \in X^k \mid f(x_1) = \cdots = f(x_k) \, \text{for} \, x_i \neq x_j, i \neq j\}.$$

Just as in the case of the double point set these sets are considerably simpler than the sets in the target formed by counting the number of preimages, the former may be non-singular in contrast to the highly singular latter. In effect, the multiple point spaces act as a resolution of the image.

There exist maps $\varepsilon_{i,k}; D^k(f) \to D^{k-1}(f)$ induced from the natural maps $\tilde{\varepsilon}_{i,k} : X^k \to X^{k-1}$ given by dropping the i^{th} coordinate from X^k. There also exists maps $\varepsilon_k : D^k(f) \to Y$ given by $\varepsilon_k(x_1, \dots, x_k) = f(x_1)$.

We will now officially define the multiple point spaces in the image that we have mentioned above.

Definition 9.4 *The k^{th}* image multiple point space*, denoted $M_k(f)$, is the space $\varepsilon_k(D^k(f))$.*

As stated earlier, the spaces $M_k(f)$ can be highly singular compared to $D^k(f)$, in the sense that $D^k(f)$ could be non-singular but $M_k(f)$ could have non-isolated singularities.

Example 9.5 *Let* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *be the Whitney umbrella* $f(x, y) = (x, xy, y^2)$. *Then,*

$$\begin{aligned} D^2(f) &= \text{closure}\{(x_1, y_1, x_2, y_2) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid (x_1, x_1y_1, y_1^2) = (x_2, x_2y_2, y_2^2); \\ &\qquad (x_1, y_1) \neq (x_2, y_2)\} \\ &= \{(0, y_1, 0, -y_1) \in \mathbb{R}^4\}. \end{aligned}$$

From this we can see that $D^2(f)$ *is a manifold – in fact, a line – and that* $M_2(f)$ *is not a manifold (although one could count it as a manifold with boundary).*

Also, we can see clearly that S_2 *acts on* $D^2(f)$.

Morin described stable map germs $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ where $n < p$ and f has corank 1. He stated that f is equivalent to a map of the form,

$$\begin{aligned} &(x_1, \dots, x_s, u_1, \dots, u_{l-2}, w_{1,1}, w_{1,2}, \dots, w_{p-n+2,l}, y) \\ &\mapsto (\underline{x}, \underline{u}, \underline{w}, y^l + \sum_{i=1}^{l-2} u_i y^i, \sum_{i=1}^{l-1} w_{1,i} y^i, \dots, \sum_{i=1}^{l-1} w_{p-n+2,i} y^i), \end{aligned}$$

where l is the multiplicity of the germ.

Mather proved in [47] that stable multi-germ maps are constructed from stable mono-germs with these mono-germs meeting transversally. Thus we have an explicit description of a map $f : N \to P$ if it is such that $\dim N < \dim P$ and is stable and corank 1. (Recall that a map is corank 1 if at each point the differential has corank at most 1. That is, the map could in theory, in this case, be an immersion at a point.)

From this explicit description we can produce a large number of examples via the following theorem.

Theorem 9.6 ([43]) *Suppose that* $f : N \to P$ *is such that* $\dim N < \dim P$ *and that the singularities of* f *are stable and corank* 1. *Then,* $D^k(f)$ *is non-singular.*

The proof of this is given in [43]. Furthermore, they give a method for calculating local defining equations for $D^k(f)$ using determinants. A simpler proof of the theorem (and one that holds in the smooth case as well) is to reduce to the problem of normal forms for the singularities considered. This is possible since, for a multi-germ f, if there is a local change of coordinates in source and target to produce f', then $D^k(f)$ and $D^k(f')$ are locally diffeomorphic.

Now, as discussed, corank 1 multi-germs have been classified by Morin, [50], for mono-germs, and Mather, [47], for multi-germs. Using Morin's description for mono-germs and the Marar-Mond description for defining equations it is straightforward to calculate that $D^k(f)$ is non-singular. In the case of multi-germs, we observe from Mather's classification that the multi-germ occurs as the trivial unfolding of a some stable mono-germ. Thus since we know the multiple point space is non-singular at the mono-germ it must be in some neighbourhood.

In fact, it is possible to prove a converse: If f is corank 1 and the kth multiple point spaces are non-singular of dimension $k \dim N - (k-1) \dim P - 1$ for all k, then f is stable.

For corank 2 stable maps, unfortunately, $D^2(f)$ can be singular (and probably is so in general).

Marar and Mond have a stronger result in [43] involving the restriction of $\tilde{D}^k(f)$ to fixed point sets of the action of S_k on $\mathbb{C}^{n-1} \times \mathbb{C}^k$. Also, they deal with the case of isolated instabilities. Here they show that the $D^k(f)$ can have isolated singularities, which furthermore are complete intersections. It is this which is key to later results: For isolated instabilities the multiple point spaces are isolated complete intersection singularities and hence we if we perturb the instability to produce a stable map, then the ICIS are perturbed to their Milnor fibres (as the multiple point spaces of a stable map are non-singular by the above).

It is easy to see that given a map we can associate lots of invariants to it by taking invariants of the multiple point spaces.

10 The Image Computing Spectral Sequence

Alternating Homology of a Complex

As stated earlier the multiple point spaces act as resolution of the image. However, since the triple point space is the double point space of the natural map $D^2(f)$ to X given by projection, it is obvious that if we use the homology of the multiple point spaces to give us the homology of the image that we get too much information. It turns out that we need to look at the alternating homology of the multiple point spaces. This is where we exploit the symmetry of the multiple point spaces. The group, S_k, of permutations on k objects acts naturally on $D^k(f)$ by permutation of copies of X^k. The alternating homology is the homology of the subcomplex of chains that alternate, i.e., are anti-symmetric with respect to S_k.

Let us put the details to this. Denote by sign the natural sign representation for S_k. The space $Z \subset X^k$ is called S_k*-cellular* if it is S_k-homotopy equivalent to a cellular complex. That is, there is a homotopy equivalence (respecting the action) to a complex of cells upon which S_k acts cellularly. (This latter means that cells go to cells and if a point of a cell is fixed by an element of S_k, then the whole cell is fixed by the element.) Whitney stratified spaces for which the strata are S_k-invariant can be triangulated to respect the action and hence have a cellular action.

Definition 10.1 *Let*

$$\mathrm{Alt}_{\mathbb{Z}} = \sum_{\sigma \in S_k} \mathrm{sign}(\sigma)\sigma.$$

We define alternating homology by applying this operator.

Definition 10.2 *The* alternating chain complex of Z, $C_*^{alt}(Z;\mathbb{Z})$ *is defined to be the following subcomplex of the cellular chain complex,* $C_n(Z;\mathbb{Z})$*, of* Z*,*

$$C_n^{alt}(Z;\mathbb{Z}) := \mathrm{Alt}_{\mathbb{Z}}\, C_n(Z;\mathbb{Z}).$$

The elements of $C_n^{alt}(Z;\mathbb{Z})$ are called *alternating* or *alternated chains.*

There is an alternative way to define or a useful way to calculate $C_n^{alt}(Z;\mathbb{Z})$:

$$C_n^{alt}(Z;\mathbb{Z}) \cong \{c \in C_n(Z;\mathbb{Z}) \mid \sigma c = \mathrm{sign}(\sigma)c \text{ for all } \sigma \in S_k\}.$$

Now we just need to apply the homology functor to this subcomplex to get alternating homology.

Definition 10.3 *The* alternating homology of Z*, denoted* $H_*^{alt}(Z;\mathbb{Z})$*, is defined to be the homology of* $C_*^{alt}(Z;\mathbb{Z})$*.*

Note that in [18] $H_i^{alt}(Z;\mathbb{Z})$ denotes the alternating part of integral homology. However, our notation is more in keeping with traditional notation in homology.

If we wish to define alternating homology over general coefficients then we may do so in the usual way by tensoring $C_*^{alt}(Z;\mathbb{Z})$ by the coefficient group.

Example 10.4 *Suppose* $T = S^1 \times S^1$ *denotes the standard torus. Then* S_2 *acts on* T *by permutation of the copies of* S^1. *Let* Z *be the points* $(z, z+\pi) \in T$, *then* Z *is just a circle with antipodal action. We can give* Z *a cellular structure by choosing two antipodal points* p_1 *and* p_2 *as 0-cells and then the complement of these points will form two 1-cells,* e_1 *and* e_2, *upon which* S_2 *acts by permutation and whose orientation we induce from an orientation of the circle. Then* $\sigma(e_1) = e_2$ *and* $\sigma(e_2) = e_1$, *where* σ *is the non-trivial element of* S_2.

The group $C_0^{alt}(Z;\mathbb{Z})$ *is generated by* $c_0 = p_1 - p_2$ *and* $C_0^{alt}(Z;\mathbb{Z})$ *is generated by* $c_1 = e_1 - e_2$. *The boundary of* c_1 *is* $-p_1 + p_2 - p_1 + p_2 = -2(p_1 - p_2)$. *Therefore* c_1 *is not a cycle and* $2(p_1 - p_2)$ *is a boundary, hence*

$$\begin{aligned} H_0^{alt}(Z;\mathbb{Z}) &= \mathbb{Z}_2, \\ H_1^{alt}(Z;\mathbb{Z}) &= 0. \end{aligned}$$

An alternating homology group is not a subgroup of ordinary homology as the example shows: $H_0^{alt}(D^2(f);\mathbb{Z}) = \mathbb{Z}_2$ is not a subgroup of $H_0(D^2(f);\mathbb{Z}) = \mathbb{Z}$.

Our fundamental example for alternating homology is $D^k(f) \subset X^k$. Suppose that, for $k > 1$, the S_k-action on $D^k(f)$ is cellular. Then we can define alternating homology for $D^k(f)$.

Example 10.5 *Let* $f : B^2 \to \mathbb{RP}^2$ *be the quotient map that maps the unit disc* B^2 *to real projective space by antipodally identifying points on the boundary of the disc. Then* $D^2(f) \subset B^2 \times B^2$ *is just the circle in Example 10.4 and so* $H_0^{alt}(D^2(f);\mathbb{Z})$ *has the alternating homology of that example. The set* $D^3(f)$ *is empty.*

Let $D^k(f)^g$ denotes the fixed point set in $D^k(f)$ of the element $g \in S_k$, and let $\chi^{alt}(D^k(f)) = \sum_i (-1)^i \dim_{\mathbb{Q}} \operatorname{Alt} H_i(X;\mathbb{Q})$ denote the alternating Euler characteristic.

Lemma 10.6 ([31]) *Assume that* S_k *induces a cellular action on* $D^k(f)$. *Then,*

$$\chi^{alt}(D^k(f)) = \frac{1}{k!} \sum_{g \in S_k} \operatorname{sign}(g) \chi(D^k(f)^g),$$

provided that each $\chi(D^k(f)^g)$ *is defined.*

A similar theorem is stated in section 3 of [19] in the case that each $D^k(f)^g$ is the Milnor fibre of an isolated complete intersection singularity, however, their result as stated is false.

This result is very useful for low dimensional cases for calculating the homology of an image since, as we shall see in a moment, that the Euler characteristic of the image is the alternating sum of the alternating Euler characteristics of the multiple point spaces.

Example 10.7 *Let us see in a simple example how the alternating homology of multiple point spaces arises in the computation of the homology of an image.*

Consider a map $f : M \to P$ *such that* $D^3(f) = \emptyset$. *Then, there is a homeomorphism of* $D^2(f)$ *onto its image under the natural projection map* $D^2(f)$ *to* M. *We can see that there is a surjective map* $C_*(M)$ *to* $C_*(f(M))$. *The kernel of this is the complex of alternating chains.*

Figure 8 shows an example where two hemispheres are glued along their edges to produce a sphere. One can see that in this case there is an inclusion of the double point set into M. *Here the double point set is two circles and these can given an orientation arising from the equator in the circle. At the level of chains we can*

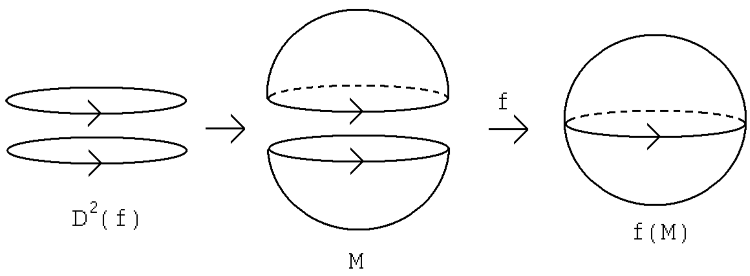

Figure 8: Example of spaces used in the short exact sequence

see that the symmetric chain maps to the chain given by the equator. Hence the alternating chains must be in the kernel of the map $C_*(M)$ to $C_*(f(M))$.

Thus, in general, we get a short exact sequence of complexes:

$$0 \to C_*^{alt}(D^2(f)) \to C_*(M) \to C_*(f(M)) \to 0,$$

and this naturally leads to a long exact sequence:

$$\cdots \to H_i^{alt}(D^2(f)) \to H_i(M) \to H_i(f(M)) \to H_{i-1}^{alt}(D^2(f)) \to \dots$$

So we have a long exact sequence that relates the homology of the image to the source and its double point space.

The Spectral Sequence

We saw that for maps with (at worst) double points we had a long exact sequence relating the homology of the image to the homology of the domain and the alternating homology of the double point space. It should therefore come as no surprise that to generalize the relation arising from this long exact sequence we need a spectral sequence.

In this section we describe such a spectral sequence that relates the homology of the image to the alternating homology of the multiple point spaces.

Theorem 10.8 *Suppose $f : X \to Y$ is a finite and proper continuous map, such that $D^k(f)$ has the S_k-homotopy type of an S_k-cellular complex for all $k > 1$ and $M_k(f)$ has the homotopy type of a cellular complex for all $k > 1$. Then there exists a spectral sequence*

$$E^1_{p,q} = H_q^{alt}(D^{p+1}(f); \mathbb{Z}) \implies H_{p+q+1}(f(X); \mathbb{Z}).$$

The differential is the naturally induced map

$$\varepsilon_{1,k_*} : H_i^{alt}(D^k(f); \mathbb{Z}) \to H_i^{alt}(D^{k-1}(f); \mathbb{Z}).$$

The proof of this is given in [30].

Example 10.9 *We have already calculated the alternating homology of the multiple point spaces for the map $f : B^2 \to \mathbb{RP}^2$ of Example 10.5. The image computing spectral sequence for f is given below.*

H_2^{alt}	0	0	0
H_1^{alt}	0	0	0
H_0^{alt}	$\mathbb{Z}$	$\mathbb{Z}_2$	0
	B^2	$D^2(f)$	$D^3(f)$

All the differentials of this sequence must be trivial and so the sequence collapses at E^1 and since there are no extension difficulties we can read off the homology of the image of f, i.e., the real projective plane.

Example 10.10 *We can use the sequence to prove Theorem 9.1 of Izumiya and Marar*

We can triangulate $f(N)$ with the cross caps and triples among the vertices and so that the image of D^2 is a subcomplex. Since f is proper and finite we can pull back the triangulation of $f(N)$ to give one for N. Thus we have a finite, proper, surjective map of CW–complexes and so we can apply the image computing spectral sequence.

The alternating homology of D^1 is just the ordinary homology of N and as there are no quadruple points the alternating homology of D^k is trivial for $k > 3$.

The triple point set D^3 is just six copies of the triple points of the source and we can see that these form alternating zero chains, one for each triple point. So $H_i^{alt}(D^3) = \mathbb{Z}^{T(f)}$ for $i = 0$ and zero otherwise.

Each normal crossing of two sheets has a cross cap at each end or is a circle and so we can pair the cross caps. The two sheeted normal crossings meet at triple points. This means that D^2 has $C(f)$ points in the diagonal from which come two 1-cells which permute under the action of S_2 on D^2. So any particular 1-cell will join a pair of cross caps and if any two 1-cells cross then they cross at a triple point. The only other parts of D^2 are μ pairs of circles.

This allows us to calculate the alternating homology of D^2. We have

$$H_i^{alt}(D^2) = \begin{cases} \mathbb{Z}^{C(f)/2} \oplus \mathbb{Z}^{\mu}, & \text{for } i = 1, \\ \mathbb{Z}^{\mu}, & \text{for } i = 0, \\ 0, & \textit{otherwise.} \end{cases}$$

We can work out the Euler characteristic of the limit of a spectral sequence from the Euler characteristic of any level, (provided of course that the terms are finitely generated). So,

$$\begin{aligned} \chi(f(N)) &= \chi(E_\infty^{p,q}) \\ &= \chi(E_1^{p,q}) \\ &= \chi^{alt}(D^1) - \chi^{alt}(D^2) + \chi^{alt}(D^3) \\ &= \chi^{alt}(D^1) - [\mu - (\mu + C(f)/2)] + T(f) \\ &= \chi(N) + C(f)/2 + T(f), \end{aligned}$$

where χ^{alt} means the Euler characteristic of the alternating homology.

It should be remarked that if we can work out how the spectral sequence collapses and there are no extension problems then it may give more information than just the Euler characteristic.

Final remarks

The image computing spectral sequence has not been sufficiently applied and there are many unexplored areas in which it could be used. Consider the case of the quotient space given by the action of a finite group G on a set X. The quotient map $\pi : X \to X/G$ is finite and in many important case will be continuous. The sequence has not been investigated in this case, not even to see the relation with the classic theorems on quotients of spaces.

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