How to Think Like a Mathematician Solutions to Exercises

September 17, 2009

The following are solutions to exercises in my book *How to Think Like a Mathematician*.

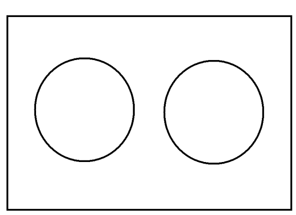
Chapter 1

Exercises 1.10 (i) 5 (ii) 3 (iii) 4 (iv) 0 (v) Infinite (vi) 2 (vii) 2 (viii) 3.
Exercises 1.20 (i) $\{-1, 1, 2, 3, 4, 5, 7\}$ (ii) $\mathbb{Z}$.
Exercises 1.23 (i)(a) $\{0, 1, 2, 3, 4, 5\}$, (b) $\emptyset$, (c) $\{0, 1, 5\}$. (ii) $\mathbb{Z}$, $\emptyset$, $\mathbb{Z}$.
Exercises 1.33 My intention was that the domain would be the largest in $\mathbb{R}$. (i) $\mathbb{R}\backslash\{(5-\sqrt{13})/2,\ (5+\sqrt{13})/2\}$, (ii) $\mathbb{R}$, (iii) Many answers are possible. Hint: We can always find a line between two points in the plane so we could use a linear function such as $f(x) = ax+b$ where a and b are determined by using the equations $f(-1) = 5$ and $f(3) = -2$. One you have mastered the idea behind this try finding a quadratic whose graph passes through the points. Then move on to functions involving higher orders or sines, cosines and the exponential function.
Exercises 1.34 (i) $A \cap B = \{2\}$, $A \cup B = \{0, 2, 3, 4, 5, 6, 7, 8, 10\}$ (or $X\backslash\{1, 9\}$ if you prefer), $A\backslash B = \{0, 4, 6, 8, 10\}$, $B\backslash A = \{3, 5, 7\}$, $A \times B$ has 24 elements (which is a lot to type), $X \times A$ has 66 elements, $A^c = \{1, 3, 5, 7, 9\}$, $B^c = \{0, 1, 4, 6, 8, 9, 10\}$.

(ii) The quadratic has roots 2 and 7 so the union is $\{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the intersection is $\{7\}$.

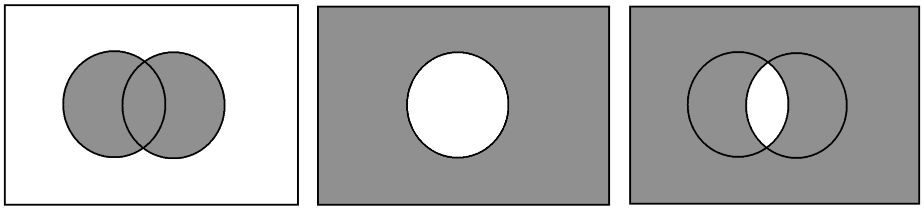
(iii) The precise answers will depend on the sets you take. You should always get equalities of sets except in (d) and (f). (This does not exclude the situation where your examples give equalities in (d) and (f). However, there exist examples where they don't have equality.)

(iv)(a)



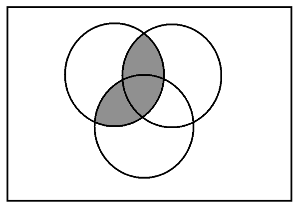
A and B disjoint.

(b)



$A \cup B$ $\quad$ A^C $\quad$ $(A \cap B)^c$

(c) An example of from (i):



$(A \cap B) \cup (A \cap B)$ or $A \cap (B \cup C)$.

(v) Not applicable.

Chapter 2

Solutions not applicable to this chapter.

Chapters 3 and 4

The solutions to these chapters will depend on your personal answers and writing style.

Exercises 3.2 (i) Don't forget to explain what $a, b, c, \alpha, \beta, \gamma$ and h are and explain what are assumptions and what are conclusions.

(ii) There are some mathematical mistakes. The second line is actually $f'(x)$ yet the student has equated it to f. Also they say (taking into account the previous comment) that $f'(x) = 6x^2 - 24x + 18$ implies that $x = 1$ and 3. This is not true, it is the equation $f'(x) = 0$ that gives us the values for x. Equality signs are missing in the calculation of x. Some notation is missing. When finding maxima and minima we put the values of x into the expression for the second derivative and so we should really have

$$\left.\frac{d^2y}{dx^2}\right|_{x=1} \text{ etc, rather than just } \frac{d^2y}{dx^2}.$$

Chapter 5

The point of this chapter is to get you thinking about solving problems and so giving the answers would defeat the purpose of the chapter. Keep coming back to them! However, I should say that (v) and (vi) are classics and so you should be able to find some answers with a web search.

Chapter 6

Exercises 6.10 Both truth tables should have the following form

A	B	$\sim$
F	F	T
F	T	F
T	F	F
T	T	F

Exercises 6.11 (i) Statements: (a), (d) and (e). Statement (d) is true but the truth or otherwise cannot be known for (a) and (e). (Well, (e) will be known if you are reading this after 2089.)

(ii) The truth table should be

A	B	(a)	(b)	(c)	(d)	(e)	(f)
F	F	T	T	T	T	T	F
F	T	T	F	T	F	T	F
T	F	T	F	T	T	T	F
T	T	F	F	F	T	T	F

(iii)(a) A is false and B is true. (b) A is true and B is false. (c) A is false and B is false. (d) A is false or B is false.

(iv) The truth tables look like the following

A	B	C	(a)	(b)	(c)
F	F	F	F	F	F
F	F	T	F	T	T
F	T	F	F	F	F
F	T	T	F	T	F
T	F	F	F	F	F
T	F	T	T	T	F
T	T	F	T	T	F
T	T	T	T	T	F

(v)(a) The truth table can be constructed as follows:

A	not A	A or (not A)
F	T	T
T	F	T

(b) not(A and not A)

(c) Yes, it is but maybe I should have put this question in one of the chapters on implications.

(d) The truth table looks like

A	not A	A and (not A)
F	T	F
T	F	F

Chapter 24

Exercises 24.10 (i) Similar to Example 24.4.

(ii) Similar to Example 24.7.

(iii) Similar to Example 24.6.

(iv) We'll prove that 17 divides $3^{4n}+4^{3n+2}$ for all $n \in \mathbb{N}$ as a worked example. (If this wasn't an exercise in a chapter on induction, then, because of the $n \in \mathbb{N}$ part, alarm bells should ring in your head and you should automatically think 'I'll use induction'.)

Initial case: We want to show that 17 divides $3^{4n} + 4^{3n+2}$ when $n = 1$. We have that

$$\begin{aligned} 3^{4n} + 4^{3n+2} &= 3^{4\times 1} + 4^{3\times 1+2} \\ &= 3^4 + 4^5 \\ &= 81 + 1024 \\ &= 1105 \\ &= 17 \times 65. \end{aligned}$$

So 17 divides $3^{4n} + 4^{3n+2}$ when $n = 1$.

Induction step: Now suppose that 17 divides $3^{4n} + 4^{3n+2}$ when $n = k$ for some particular k, i.e.,

$$3^{4k} + 4^{3k+2} = 17m \text{ for some } m \in \mathbb{Z}.$$

Let's consider the expression for $n = k + 1$:

$$\begin{aligned} 3^{4(k+1)} + 4^{3(k+1)+2} &= 3^{4k+4} + 4^{3k+5} \\ &= \left(3^{4k}\right) 3^4 + 4^{3k+5} \\ &= \left(17m - 4^{3k+2}\right) 3^4 + 4^{3k+5}, \\ &\qquad\qquad \text{by the inductive hypothesis,} \\ &= 17m \times 3^4 - 4^{3k+2} \times 3^4 + 4^{3k+5} \\ &= 17m \times 3^4 - 4^{3k+2} \times 3^4 + 4^{3k+2} \times 4^3 \\ &= 17m \times 3^4 - 4^{3k+2} \times \left(3^4 - 4^3\right) \\ &= 17m \times 3^4 - 4^{3k+2} \times (81 - 64) \\ &= 17m \times 3^4 - 4^{3k+2} \times 17 \\ &= 17\left(m \times 3^4 - 4^{3k+2}\right). \end{aligned}$$

So 17 divides $3^{4(k+1)} + 4^{3(k+1)+2}$ as required.

We have shown that the statement for $n = k$ implies that the statement for $n = k + 1$ is true. Therefore by the Principle of Mathematical Induction the statement is true for all $n \in \mathbb{N}$. That is,

$$17 \text{ divides } 3^{4n} + 4^{3n+2} \text{ for all } n \in \mathbb{N}.$$

(v) Hint: $\sin(n + 1)x$ can be taken to be the 'complicated' side as we can expand it to something else. Also $-1 \leq \cos\theta \leq 1$.

(vi) Hint: In the induction step we can show that

$$\begin{aligned}(x+y)(x+y)^k &= (x+y)\sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^r \\ &= \sum_{r=0}^{k}\binom{k}{r}x^{k-r+1}y^r + \sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^{r+1}\end{aligned}$$

At this point a number of students start writing out the summations in full so that they rearrange and gather them together. I.e., they would like to gather the terms $x^a y^b$ from the two summations. We can do this by 'changing the variable' in one of the summations.

Let $s = r + 1$, so when $r = 0$, $s = 1$ and when $r = k$, $s = k + 1$. We also have $r = s - 1$. Then the second summation can be changed:

$$\begin{aligned}\sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^{r+1} &= \sum_{s=1}^{k+1}\binom{k}{s-1}x^{k-(s-1)}y^{(s-1)+1} \\ &= \sum_{s=1}^{k+1}\binom{k}{s-1}x^{k-s+1}y^{s}.\end{aligned}$$

Now here's the part that upsets some people. The summation does not depend on the name of the variable used in the summing and so I can change the s to r. But $s = r - 1$ they say, we can't take $s = r$. The answer is that this r is different to the previous r. Sounds confusing but mathematicians do this type of change all the time!

We have

$$\sum_{s=1}^{k+1}\binom{k}{s-1}x^{k-s+1}y^{s} = \sum_{r=1}^{k+1}\binom{k}{r-1}x^{k-r+1}y^{r}.$$

In conclusion this means we have shown that

$$\sum_{r=0}^{k}\binom{k}{r}x^{k-r}y^{r+1} = \sum_{r=1}^{k+1}\binom{k}{r-1}x^{k-r+1}y^{r}.$$

Thus we get that

$$(x+y)(x+y)^k = \sum_{r=0}^{k}\binom{k}{r}x^{k-r+1}y^r + \sum_{r=1}^{k+1}\binom{k}{r-1}x^{k-r+1}y^{r}.$$

From here you can separate out the term $r = 0$ in the first summation and $r = k + 1$ in the second so that you can add the two remaining parts of the summations together.

In my original notes for the solution to this problem I added together $\binom{k}{r}$ and $\binom{k}{r-1}$ by using the definition in terms of factorials. Instead you can use the identity from the statement of exercise (viii), which amounts to the same thing.

(vii) Change $n^2 - 1$ to $(2n-1)^2 - 1$ as described on page 172. In this exercise we can also prove the statement by the direct method. It is easy to calculate that $(2n-1)^2 - 1 = 4n(n-1)$. Now for any n either n or $n-1$ is even and hence so is their product. This means that $n(n-1) = 2m$ for some $m \in \mathbb{Z}$.

Therefore, $(2n-1)^2 - 1 = 4 \times 2m = 8m$ and we can conclude that the number is divisible by 8.

(viii) Similar to (vi).

(ix) Similar to Example 24.4 and exercise (i) but with more complicated algebraic manipulation. A surprising result, don't you think? That the square of the sum of the first n numbers is equal to the sum of the cubes of the first n numbers is surprising to me.

(x) This exercise is incorrect. The statement is not even complete. See the corrections on the website.

(xi) This question is about counting subsets and to some extent about how you organize your writing of mathematics. In the inductive step we consider that X has $k+1$ elements and so it contains a subset Y of k elements so that $X = Y \cup \{x\}$ for some $x \in X$. By the inductive hypothesis Y has 2^k subsets. As Y is a subset of X, it must be true that X has these as subsets too. Next we can see that if Z is a subset of Y, then $Z \cup \{x\}$ will be a subset of X that we haven't already counted. There should therefore be 2^k of these. Thus X should have 2×2^k subsets. The aim of the exercise is to rigorously count these subsets and to write your answer so that is comprehensible. This is the organizational part, you need to give good notation so that the counting process is easy/clear.

(xii)(a) The inductive hypothesis is $x^k - 1 = m(x-1)$ for some $m \in \mathbb{Z}$ so $x^k = m(x-1) + 1$. The rest is just algebra. (b) The formula is

$$\frac{x^n - 1}{x - 1} = x^n + x^{n-1} + \cdots + x + 1$$

and the latter can be written succinctly as $\sum_{i=0}^{n} x^i$.

(xiii) We have

$$2^n < 2^{n-1} \iff 2.2^{n-1} < 2^{n-1} \iff 2 < 1.$$

So the statement is false. It is important that we use $\iff$, see Chapter 21 on common mistakes. The induction step should be straightforward.

(xiv) The induction step involves dealing with the two cases, $\forall$ and $\exists$, separately.

(xv) This one should be straightforward as I think that neither side of the inequality is more complicated than the other so it doesn't matter which you pick to work on.

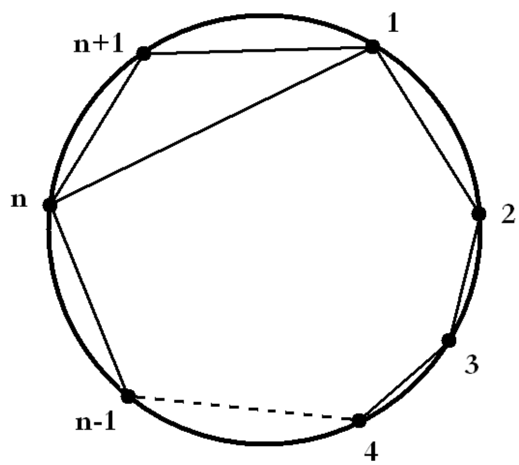
(xvi) The algebra for this is the hardest part so don't worry if your answer is long. You may need to use the formula in (xii)(b). The problem is about comparing the arithmetic and geometric means of a collection of numbers.

Chapter 25

Exercises 25.2 (i) One answer is that the proof goes wrong for $k < 7$ simply because those cases are not true. I.e., the initial case is missing. One could also argue that – sometimes – the induction step is false when $k < 7$. That is, in the proof, $k^2 + 3k \leq k^2 + k \times k$ is true for $k \geq 7$ but in fact it is also true for $k \geq 3$ while not true for $k = 1$ and 2.

(ii) Similar to (i). The inductive step is true for all $k \geq 1$.

(iii) Hint:



Exercises 25.4 (i) The two initial cases are easy. The induction step is more complicated. Here's how I would attempt it. I am trying to show you that, sometimes, solving a problem involves dead ends and refinement and that it may not be obvious, after it is written up, where the answer came from.

Let's begin the proof of the induction step: Assume that $A(k)$ and $A(k-1)$ are true, i.e., $x_k < \left(\frac{7}{4}\right)^k$ and $x_{k-1} < \left(\frac{7}{4}\right)^{k-1}$. Then

$$
\begin{aligned}
x_{k+1} &= x_k + x_{k-1}, \text{ by definition,} \\
&< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}, \text{ by the inductive hypothesis,} \\
&< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^k, \text{an obvious step towards grouping terms,} \\
&= 2\left(\frac{7}{4}\right)^k \\
&> \frac{7}{4}\left(\frac{7}{4}\right)^k .
\end{aligned}
$$

So unfortunately, at the end I don't get the 'less than $\left(\frac{7}{4}\right)^{k+1}$ I would like. The problem arises in the obvious step towards grouping terms where I used $\left(\frac{7}{4}\right)^{k-1} < \left(\frac{7}{4}\right)^k$. Taking the RHS term has led to something bigger than I would like, so I should take something smaller on the RHS.

So let's try again:

$$\begin{aligned}
x_{k+1} &= x_k + x_{k-1}, \text{ by definition,}\\
&< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}, \text{ by the inductive hypothesis,}\\
&< \left(\frac{7}{4}\right)^k + a\left(\frac{7}{4}\right)^k, \text{ when } 1 < \frac{7}{4}a,\\
&= (1+a)\left(\frac{7}{4}\right)^k.
\end{aligned}$$

So if I took $a = 3/4$, (and this is OK as $1 < \frac{7}{4} \times \frac{3}{4}$), then I would get

$$(1+a)\left(\frac{7}{4}\right)^k = \left(1+\frac{3}{4}\right)\left(\frac{7}{4}\right)^k = \left(\frac{7}{4}\right)\left(\frac{7}{4}\right)^k = \left(\frac{7}{4}\right)^{k+1}.$$

And this is what I wanted!

Now, obviously this is just rough work and I wouldn't hand in all this working out if I were submitting it as part of an assignment. Instead I would just do the following:

$$\begin{aligned}
x_{k+1} &= x_k + x_{k-1}, \text{ by definition,}\\
&< \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1}, \text{ by the inductive hypothesis,}\\
&< \left(\frac{7}{4}\right)^k + \frac{3}{4}\left(\frac{7}{4}\right)^k\\
&= \left(1+\frac{3}{4}\right)\left(\frac{7}{4}\right)^k\\
&= \left(\frac{7}{4}\right)\left(\frac{7}{4}\right)^k\\
&= \left(\frac{7}{4}\right)^{k+1}.
\end{aligned}$$

Hence $A(k+1)$ is true.

Notice that in the above, in the polished final answer, I didn't explain where the 3/4 came from I just used it and the reader is left to verify the fairly simple observation that $\left(\frac{7}{4}\right)^{k-1} < \frac{3}{4}\left(\frac{7}{4}\right)^k$. This is another example where we cover our tracks.

(ii) We need $A(2)$ true as well or else we can't use the inductive step to prove that $A(3)$ is true. In other words we can't get started without it.

Exercises 25.6 (i) Not applicable. (ii) The proof doesn't involve induction. Try contradiction instead.

Exercises 25.7(i)(a) $\{1, 1, 2, 3, 5, 8, 13\}$,

(b) The algebraic manipulation after adding the forms for x_k and x_{k-1} gets a bit messy but stick with it, terms 'magically' disappear.

(c) We have

$$\begin{aligned}
x_{3(k+1)} &= x_{3k+3}\\
&= x_{3k+2} + x_{3k+1}, \text{ by definition,}\\
&= (x_{3k+1} + x_{3k}) + x_{3k+1}, \text{ again by definition,}\\
&= 2x_{3k+1} + 2m, \text{ by the inductive hypothesis.}
\end{aligned}$$

The other details are fairly straightforward.

In fact, it is possible to prove that $x_{3n} = \sum_{i=1}^{n} \binom{n}{i} 2^i x_i$. You may like to attempt the proof of this.

(d) The formula is $\sum_{i=1}^{n} x_{2i-1} = x_{2n}$.

(e) The formula is $\sum_{i=1}^{n} x_{2i} = x_{2n+1} - 1$.

(ii) It is easier if you simplify the $\frac{2^{k-1}}{3\times 2^{2k-2}}$ type terms first but this is not strictly necessary. Also, don't forget that

$$(-1)^{k+1} = (-1)^2(-1)^{k-1} = 1.(-1)^{k-1} = (-1)^{k-1}.$$

(iii) The factorization can be done in a number of ways (we don't need induction). I can see that 12870 is divisible by 10 and so 2 and 5 are factors. So I need to factor 1287. To do this I find the square root as it gives me the highest number that I need to check to find a factor. If there is a factor greater than the square root it must be balanced by one smaller than the square root. (Think about it.) My calculator gives $\sqrt{1287} = 3\sqrt{143}$ from which I deduce that 3^2 is a factor of 1287. I now need to factor $1287/9 = 143$. Again I use my calculator, this time it says that $\sqrt{143} \simeq 11.95$ so I need only to check divisibility by the primes from 2 to 11. However, when thinking about 11 I notice that the number 143 is of the form abc where $a + c = b$ and so I know that 143 is divisible by 11. I can work out that $143/11 = 13$. This is prime so I can now stop my factorization.

Putting this all together we get

$$12870 = 2.5.3^2.11.13.$$

The number 17836 is done in a similar manner. I can see that the number is even so I can divide by 2. The resulting number is even also and so I can divide by 2 again. Alternatively, I can see that the last two digits are divisible by 4 and so the number itself is divisible by 4. (A proof of this is required in Exercises 27.23(ix)(b)).

We have $17836/4 = 4459$. This is not even so is not divisible by 2. Again I find the square root so that I have a bound on the factors I should look for, here $\sqrt{4459} = 7\sqrt{91}$. Thus, $4459 = 7^2 \times 91$. Assuming I don't know the divisors of 91 I reach for my calculator again and get $\sqrt{91} \simeq 9.5$ so we only need check the primes 2, 3, 5, and 7 for divisibility. Obviously 2 and 5 can be ignored as 91 is not even and does not end in 0 or 5. Also, 3 can be discarded as 90 is obviously a multiple of 3. This leaves 7, a quick calculation shows that $91/7 = 13$. As this is prime we stop.

Putting it all together we get:

$$17836 = 4 \times 7^2 \times 7 \times 13 = 2^2.7^3.13.$$

Kevin Houston
School of Mathematics
University of Leeds
Leeds
UK
LS2 9JT
k.houston@leeds.ac.uk
www.maths.leeds.ac.uk/∼khouston