Vector Fields Liftable Over Stable Maps

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Outline

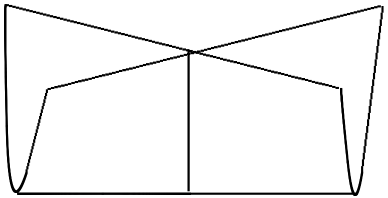
Motivation

The motivation for this study originated in generic geometry but the results have been applied in wider situations.

Suppose that we have a map from a surface to $\mathbb{R}^3$.

The generic situation is that the singularities of the map are Whitney cross-caps (umbrellas) or transverse crossings of two or three planes.

The (Whitney) cross-cap $\varphi : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ is given by $\varphi(x, y) = (x, y^2, xy)$.



- corank 1, i.e., the rank of the Jacobian drops by at most 1 at the singular point,
- stable, i.e., small perturbations give (up to diffeomorphism) the same map.

We know that we can study manifolds by studying functions, such as the height function and distance squared function (as described in the Izumiya-Ruas mini-course).

We would like to do the same on spaces given as the image of generic maps. For surfaces this means studying functions on the cross-cap.

Janet West in her 1995 PhD thesis began this classification and she published a paper on it with Bill Bruce:
Bruce and West, Functions on a crosscap, Mathematical Proceedings of the Cambridge Philosophical Society (1998)

To classify functions on the cross-cap they needed to make diffeomorphisms of the ambient space such that the cross-cap was preserved.

To generate these diffeomorphisms they needed to know the vector fields tangent to the image.

In the complex case these are equivalent to the liftable vector fields.

Liftable Vector Fields

Definition

Let f be a smooth mapping $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. A vector field ξ on $(\mathbb{K}^p, 0)$ is *liftable over f* if there is a vector field η on $(\mathbb{K}^n, 0)$ such that $df \circ \eta = \xi \circ f$. That is, the following diagram commutes

$$\begin{array}{ccc} T(\mathbb{K}^n, 0) & \xrightarrow{df} & T(\mathbb{K}^p, 0) \\ \eta \uparrow & & \uparrow \xi \\ (\mathbb{K}^n, 0) & \xrightarrow{f} & (\mathbb{K}^p, 0). \end{array}$$

In these circumstances η is called *lowerable*.

Example

For the cross-cap $\varphi(v_1, y) = (v_1, y^2, v_1 y) = (V_1, W_1, W_2)$ the following are liftable vector fields:

$$\begin{pmatrix} W_2 \\ 0 \\ V_1 W_1 \end{pmatrix}, \begin{pmatrix} -V_1 \\ 2W_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2W_2 \\ V_1^2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} V_1 \\ 2W_1 \\ 2W_2 \end{pmatrix}.$$

The corresponding lowerable vector fields are (respectively)

$$\begin{pmatrix} v_1 y \\ 0 \end{pmatrix}, \begin{pmatrix} -v_1 \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ y \end{pmatrix}.$$

These vector fields generate the module of liftable vector fields.

Example

Taking the second vector field in the list we have

$$
\begin{aligned}
d\varphi \circ \begin{pmatrix} -v_1 \\ y \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 2y \\ y & v_1 \end{pmatrix} \begin{pmatrix} -v_1 \\ y \end{pmatrix} \\
&= \begin{pmatrix} -v_1 \\ 2y^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -V_1 \\ 2W_1 \\ 0 \end{pmatrix} \circ \varphi .
\end{aligned}
$$

Minimal cross-cap

Definition

For $k \geq 2$ the *minimal cross-cap mapping of multiplicity k* is the map $\varphi_k : (\mathbb{K}^{2k-2}, 0) \to (\mathbb{K}^{2k-1}, 0)$ given by

$$\varphi_k(u_1, \dots, u_{k-2}, v_1, \dots, v_{k-1}, y) = \left(u_1, \dots, u_{k-2}, v_1, \dots, v_{k-1}, y^k + \sum_{i=1}^{k-2} u_i y^i, \sum_{i=1}^{k-1} v_i y^i \right)$$

We shall label the coordinates of the target $U_1, \dots, U_{k-2}, V_1, \dots, V_{k-1}, W_1$ and W_2, respectively.

Example

The Whitney cross-cap is $\varphi_2(v_1, y) = (v_1, y^2, v_1 y)$.

- Daniel Littlestone's task: Find liftables for φ_k.
- Holland and Mond 1999: For $\mathbb{K} = \mathbb{C}$ the module of liftable vector fields is generated by $3k - 2$ liftables.
- Thus image of φ_k is not a free divisor. Has $k - 1$ 'extra' generators. Compare with $n \geq p$ case.
- Used Singular for small k. Guessed form of lowerables. Then generalized form for liftables and lowerables. Used $d\varphi_k \circ \eta = \xi \circ \varphi_k$.

Euler vector field

One vector field was easy. The Euler vector field:

$$\xi_e = \begin{pmatrix} (k-1)U_1 \\ (k-2)U_2 \\ \vdots \\ 2U_{k-2} \\ (k-1)V_1 \\ (k-2)V_2 \\ \vdots \\ V_{k-1} \\ kW_1 \\ kW_2 \end{pmatrix}$$

The other $3k-3$ vector fields come in three families. We need some way of describing these liftables in a compact form.

$$\xi_j^f = \begin{pmatrix} A_{1,j}^f \\ \vdots \\ A_{k-2,j}^f \\ B_{1,j}^f \\ \vdots \\ B_{k-1,j}^f \\ C_{1,j}^f \\ C_{2,j}^f \end{pmatrix} \quad \begin{matrix} U_1 \\ \vdots \\ U_{k-2} \\ V_1 \\ \vdots \\ V_{k-1} \\ W_1 \\ W_2 \end{matrix}$$

$f = 1, 2, 3$ denotes the family.
$j = 1, 2, 3, \ldots, k-1$ denotes which member of the family.

First Family

Theorem (Littlestone)

For $1 \leq j \leq k-1$ the vector field given by the following components is liftable over φ_k:

$$
\begin{aligned}
A^1_{i,j} &= (k-i)(k-j)U_iU_j, \qquad 1 \leq i \leq k-2, \\
B^1_{i,j} &= k\sum_{r=1}^{i-1} U_{i+j-r}V_r - k\sum_{r=1}^{i} U_rV_{i+j-r} - (i-1)(k-j)U_jV_i \\
&\quad + kV_{i+j}W_1 - kU_{i+j}W_2, \qquad 1 \leq i \leq k-1, \\
C^1_{1,j} &= k(k-j)U_jW_1, \\
C^1_{2,j} &= -kV_jW_1 + (k-j)U_jW_2.
\end{aligned}
$$

We set $U_{k-1} = V_k = 0$, $U_k = 1$ and $U_i = V_i = 0$ for $i < 0$ and $i > k$.

Second family

Theorem (Littlestone)

For $1 \leq j \leq k-1$ the vector field given by the following components is liftable over φ_k:

$$
\begin{aligned}
A^2_{i,j} &= -k(k+i-j+1)U_{k+i-j+1}W_1 + k\sum_{r=1}^{i}(k+i-j-2r+1)U_rU_{k+i-j-r+1} \\
&\quad -j(i+1)U_{i+1}U_{k-j}, \qquad 1 \leq i \leq k-2, \\
B^2_{i,j} &= -k(k+i-j+1)V_{k+i-j+1}W_1 + k\sum_{r=1}^{i}(k+i-j-r+1)U_rV_{k+i-j-r+1} \\
&\quad -k\sum_{r=1}^{i} rU_{k+i-j-r+1}V_r - j(i+1)U_{k-j}V_{i+1}, \qquad 1 \leq i \leq k-1, \\
C^2_{1,j} &= k(k-j+1)U_{k-j+1}W_1 + jU_1U_{k-j}, \\
C^2_{2,j} &= k(k-j+1)V_{k-j+1}W_1 + jV_1U_{k-j}.
\end{aligned}
$$

Third family

Theorem (Littlestone)

For $1 \leq j \leq k-1$ the vector field given by the following components is liftable over φ_k:

$$
\begin{aligned}
A^3_{i,j} &= -k(k+i-j+1)U_{k+i-j+1}W_2 + k\sum_{r=1}^{i}(k+i-j-r+1)U_{k+i-j-r+1}V_r \\
&\quad -k\sum_{r=1}^{i} rU_r V_{k+i-j-r+1} - k(i+1)U_{i+1}V_{k-j}, \qquad 1 \leq i \leq k-2, \\
B^3_{i,j} &= -k(k+i-j+1)V_{k+i-j+1}W_2 + k\sum_{r=1}^{i}(k+i-j-2r+1)V_r V_{k+i-j-r+1} \\
&\quad -k(i+1)V_{i+1}V_{k-j}, \qquad 1 \leq i \leq k-1, \\
C^3_{1,j} &= k(k-j+1)U_{k-j+1}W_2 + kU_1 V_{k-j} \\
C^3_{2,j} &= k(k-j+1)V_{k-j+1}W_2 + kV_1 V_{k-j}.
\end{aligned}
$$

The proofs are just long calculations involving writing down a lowerable and showing that $d\varphi_k \circ \eta = \xi \circ \varphi_k$.

An example

$\varphi_3 : (\mathbb{K}^4, 0) \to (\mathbb{K}^5, 0)$: (Variables: U_1, V_1, V_2, W_1, W_2)

$$\xi_1^1 = \begin{pmatrix} 4U_1^2 \\ -3U_1V_1 + 3V_2W_1 - 3U_2W_2 \\ 3U_2V_1 - 3(U_1V_2 + U_2V_1) - 2U_1V_2 + 3V_3W_1 - 3U_3W_2 \\ 6U_1W_1 \\ -3V_1W_1 + 2U_1W_2 \end{pmatrix}.$$

However, recall $U_2 = V_3 = 0$ and $U_3 = 1$.

$$\xi_1^1 = \begin{pmatrix} 4U_1^2 \\ -3U_1V_1 + 3V_2W_1 \\ -5U_1V_2 - 3W_2 \\ 6U_1W_1 \\ -3V_1W_1 + 2U_1W_2 \end{pmatrix}$$

Liftable vector fields for $k = 3$

$$\xi_1^1 = \begin{pmatrix} 4U_1^2 \\ -3U_1V_1 + 3V_2W_1 \\ -5U_1V_2 - 3W_2 \\ 6U_1W_1 \\ -3V_1W_1 + 2U_1W_2 \end{pmatrix} \qquad \xi_2^1 = \begin{pmatrix} 0 \\ -3U_1V_2 - 3W_2 \\ 3V_1 \\ 0 \\ -3V_2W_1 \end{pmatrix}$$

$$\xi_1^2 = \begin{pmatrix} 6U_1 \\ -3V_1 \\ -6V_2 \\ 9W_1 \\ 0 \end{pmatrix} \qquad \xi_2^2 = \begin{pmatrix} -9W_1 \\ 2U_1V_2 \\ -3V_1 \\ 2U_1^2 \\ 6V_2W_1 + 2U_1V_1 \end{pmatrix}$$

$$\xi_1^3 = \begin{pmatrix} 9V_1 \\ -6V_2^2 \\ 0 \\ 9W_2 + 3U_1V_2 \\ 3V_1V_2 \end{pmatrix} \qquad \xi_2^3 = \begin{pmatrix} -9W_2 - 3U_1V_2 \\ -3V_1V_2 \\ 0 \\ 3U_1V_1 \\ 6V_2W_2 + 3V_1^2 \end{pmatrix}$$

$$\xi_e = \begin{pmatrix} 2U_1 \\ 2V_1 \\ V_2 \\ 3W_1 \\ 3W_2 \end{pmatrix}.$$

They generate Derlog

Theorem (H—)

The module of vector fields liftable over $\varphi_k : (\mathbb{C}^{2k-2}, 0) \to (\mathbb{C}^{2k-1}, 0)$ is generated by

$$\xi_e, \xi_j^1, \xi_j^2, \xi_j^3, \qquad 1 \leq j \leq k-1.$$

The module of liftable vector fields is also known as Derlog(V) where V is the image (discriminant) of the map.

Sketch of Proof

Sketch of proof:

- The proof uses a monomial ordering and a division algorithm for modules.
- Take monomial order so that W_1 and W_2 are 'leading'. E.g. Reverse lexicographic order.
- Order module of vector fields with 'Term over position'.
- For any liftable vector field ξ divide by our vector fields so that have remainder r. This is liftable.
- Division algorithm says that remainder has no W_1s and W_2s in most positions.
- Show no lowerable exists for the remainder unless the remainder was 0.

A conjecture

The image of φ_k is a hypersurface, V_k. Suppose that a reduced defining equation for this is $d_k : (\mathbb{K}^{2k-1}, 0) \to (\mathbb{K}, 0)$.

Conjecture:

$$\xi_j^f(d_k) = 0 \text{ for } f = 1, 2, 3 \text{ and } j = 1, \ldots, k-1.$$

True for $k \leq 7$ by using Singular.
Important because then

$$\mathrm{Derlog}(V_k) \cong \langle \xi_e \rangle \oplus \langle \xi_j^1, \xi_j^2, \xi_j^3 \rangle_{j=1}^{k-1}.$$

Using $\mathrm{Derlog}(V_k)$ we can calculate the $\mathcal{A}_e$-codimension of maps generated from φ_k.
Using $\langle \xi_j^1, \xi_j^2, \xi_j^3 \rangle_{j=1}^{k-1}$ we can calculate the number of vanishing cycles associated to these maps.

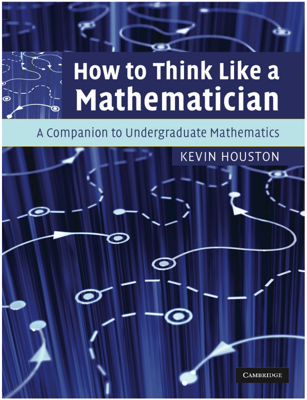
Applications

Applications:

- Classification of $\mathcal{A}_e$-codimension 1 maps.
  - Use ${}_V\mathcal{K}$-codimension of linear function (not $\mathcal{K}_V$-codimension of an immersion).
  - Tells us what generic means.
- More general classification of corank 1 maps.
  - $\mathbb{C}^3$ to $\mathbb{C}^4$.
- Topology
  - Calculate number of vanishing cycles for examples.
  - A proof of the Mond conjecture for corank 1 maps?
    - f finitely $\mathcal{A}$-determined, $\mathcal{A}_e$-codim$(f) \leq \mu_I =$number of vanishing cycles.

Shameless advert for book

My book for undergraduates ‘How to Think Like a Mathematician’ is now available:



First video is available on my YouTube channel

`http://www.youtube.com/user/DrKevinHouston`